Lattice Reduction

The Algorithmic Ant and the Sandpile

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The Algorithmic Ant and the Sandpile

Once upon a time ...

Once upon a time . . .

. . . there was an ant.

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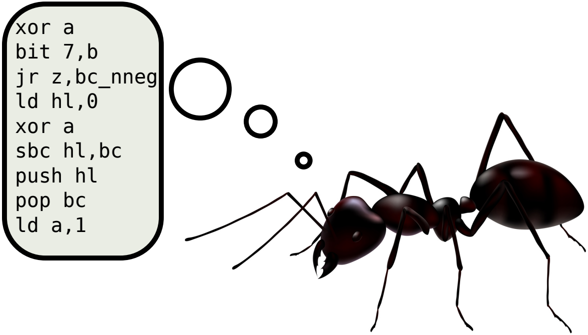
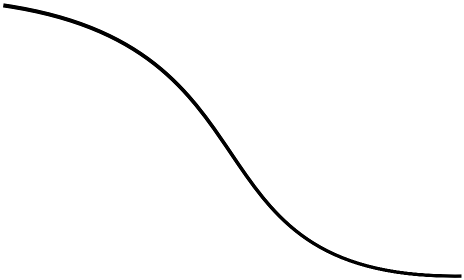


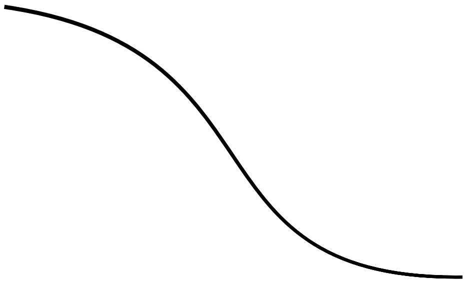
An algorithmic ant.

The Queen of ant ant,

The Queen of ant
ant,

"See this sand pile."





"I want it *flat* !"

Looking clo

the algorithmic ant ponders.

"One grain at the time,
I shall pull the sand downhill."

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So ho

But the sandpile is whimsical,

each excavation is a puzzle of it

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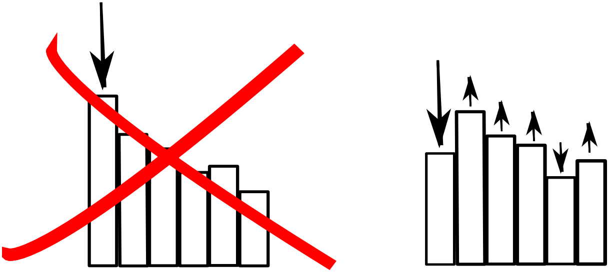

But the sandpile is whimsical,

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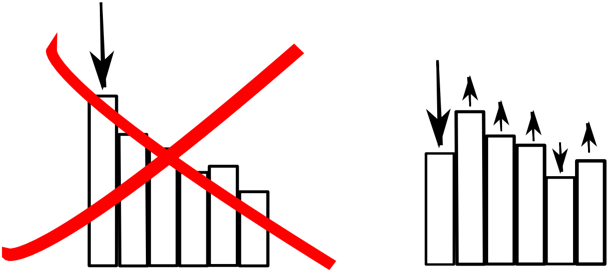
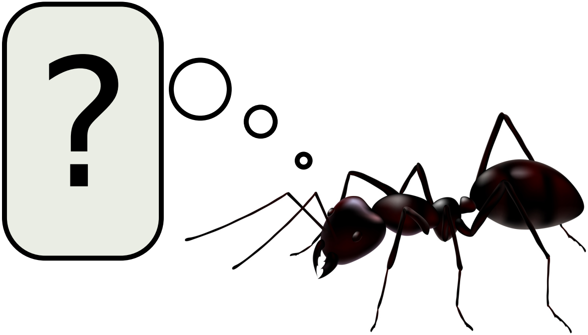


Columns are tied in my

to push one down, one must find

the right combination.

Unsure how to proceed,



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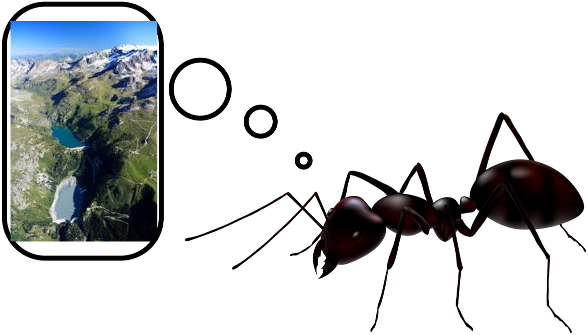
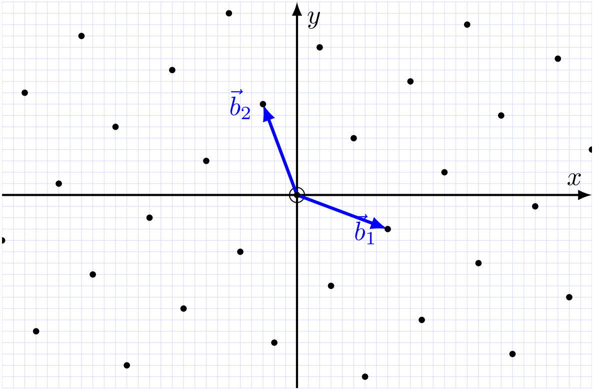


The ant joins Aussois Winter school to learn about lattice reduction.

Lattices and Bases

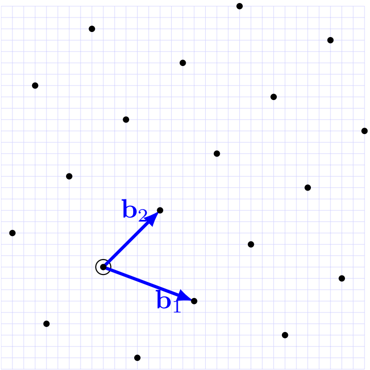
Lattices!



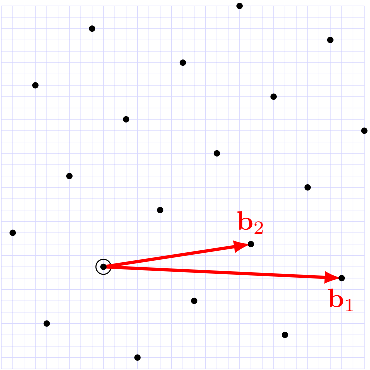
Definition

A lattice L is a discrete subgroup of a finite-dimensional Euclidean vector space.

The Good-basis/Bad-basis Routine



Good Basis $\mathbf{G}$ of L



Bad Basis $\mathbf{B}$ of L

G $\rightarrow$ **B** : easy (randomization);

B $\rightarrow$ **G** : hard (LLL, BKZ, Lattice Sieve...).

Lattice reduction: a pedantic summary

- A basis: $\mathbf{B} \in GL_n(\mathbb{R})$
- A lattice change of basis: $\mathbf{Z} \in GL_n(\mathbb{Z})$
- A lattice: $\Lambda \in GL_n(\mathbb{R})/GL_n(\mathbb{Z})$
- Lattice reduction: finding a good representative $\mathbf{M} \in GL_n(\mathbb{R})$ of the class of $\Lambda \in GL_n(\mathbb{R})/GL_n(\mathbb{Z})$

Remarks:

1. We will not achieve canonical representation.
2. We will typically want invariance by rotations $\mathbf{Q} \in O_n(\mathbb{R})$:

$$\text{if } \mathbf{B} \rightsquigarrow \mathbf{G}, \text{ then } \mathbf{QB} \rightsquigarrow \mathbf{QG}$$

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An important invariant: the Volume

For any two bases **G**, **B** of the same lattice Λ:

$$\det(\mathbf{G}\mathbf{G}^t) = \det(\mathbf{B}\mathbf{B}^t).$$

We can therefore define:

$$\mathrm{vol}(\Lambda) = \sqrt{\det(\mathbf{G}\mathbf{G}^t)}.$$

Geometrically: the volume of any **fundamental domain of** Λ.

Let $\mathbf{G}^\star$ be the Gram-Schmidt Orthogonalization of $\mathbf{G}$

$\mathbf{G}^\star$ is **not** a basis of Λ, nevertheless:

$$\mathrm{vol}(\Lambda) = \sqrt{\det(\mathbf{G}^\star \mathbf{G}^{\star t})} = \prod \|\mathbf{g}_i^\star\|.$$

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What is a "Good" basis

Recall that, independently of the basis $\mathbf{G}$ it holds that:

$$\mathrm{vol}(\Lambda) = \prod \|\mathbf{g}_i^\star\|.$$

Therefore, it is somehow equivalent that

- $\max_i \|\mathbf{g}_i^\star\|$ is small
- $\min_i \|\mathbf{g}_i^\star\|$ is large
- $\kappa(\mathbf{G}) = \max_i \|\mathbf{g}_i^\star\| / \min_i \|\mathbf{g}_i^\star\|$ is small

Good basis

$$\max \|\mathbf{b}_i^*\| \approx \min \|\mathbf{b}_i^*\|, \qquad (\text{equivalently}: \forall i, \|\mathbf{b}_i^*\| \approx \mathrm{vol}(\Lambda)^{1/n})$$

Bad basis

$$\max \|\mathbf{b}_i^*\| \gg \min \|\mathbf{b}_i^*\|$$

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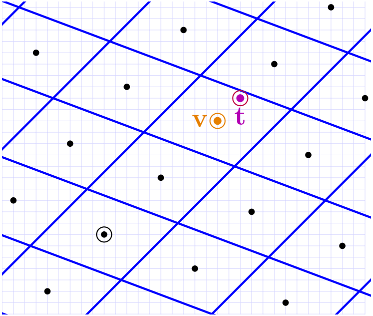
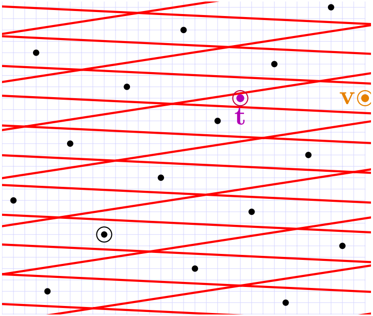
Bases and Fundamental Domains

Each basis defines a **parallelepipedic tiling**.





Round'off Algorithm [Lenstra, Babai]:

- Given a target **t**
- Find's **v** $\in L$ at the center the tile.

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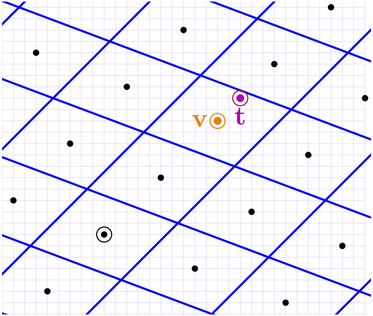

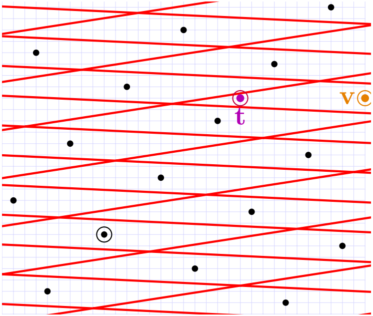

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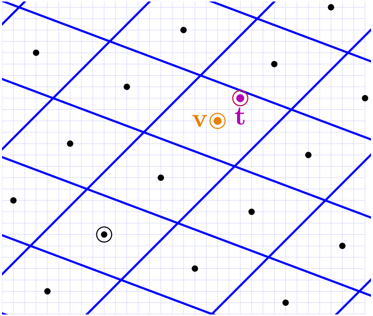

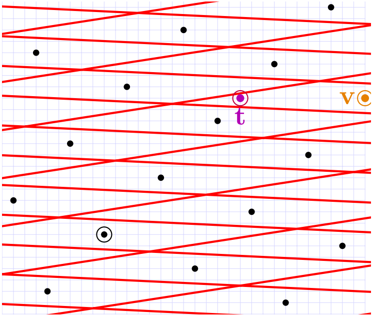

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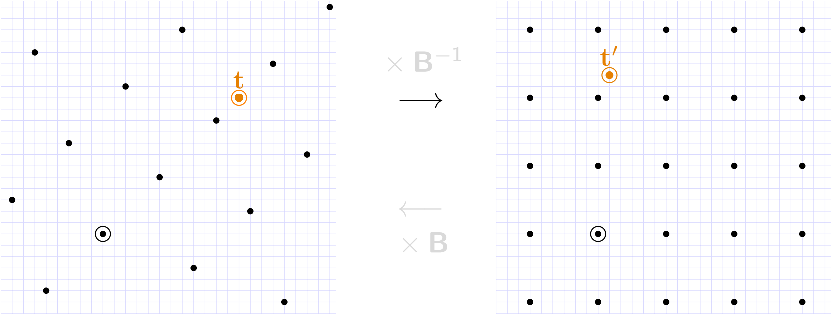

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- Use $\mathbf{B}$ to switch to the lattice $\mathbb{Z}^n$ ($\times \mathbf{B}^{-1}$)
- round each coordinate (square tiling)
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$$\mathbf{t}' = \mathbf{B}^{-1} \cdot \mathbf{t}; \quad \mathbf{v}' = \lfloor \mathbf{t}' \rceil; \quad \mathbf{v} = \mathbf{B} \cdot \mathbf{v}'$$

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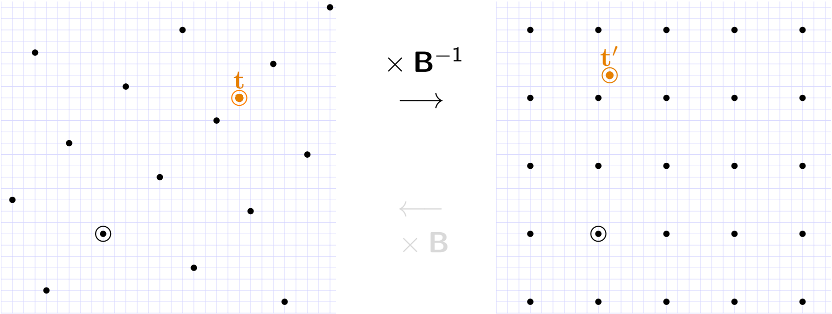

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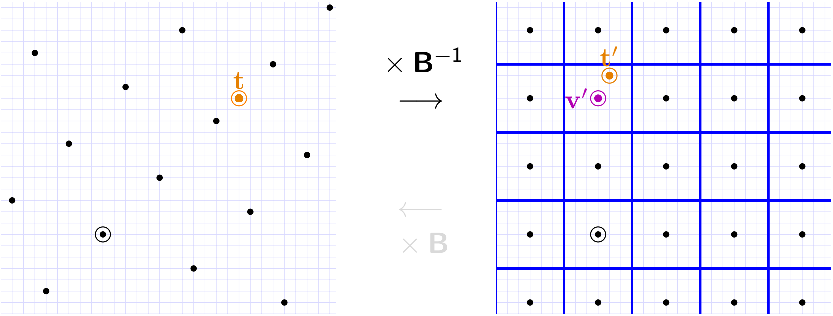

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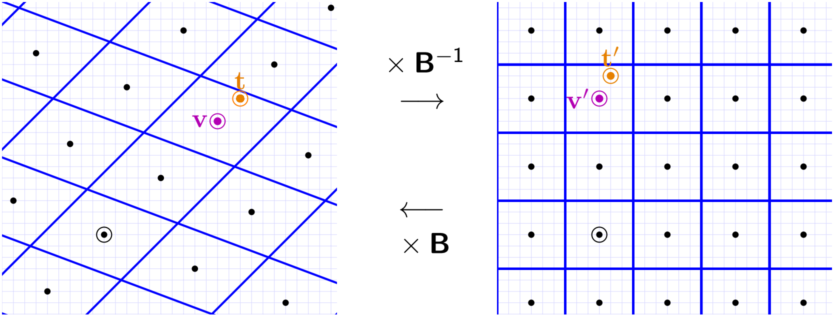

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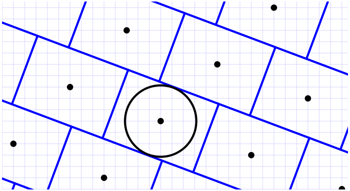
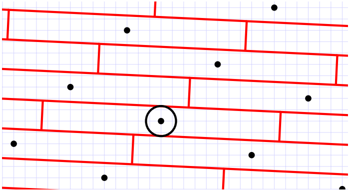
Nearest-Plane Algorithm

There is a better algorithm (NEARESTPLANE) based on Gram-Schmidt Orth. $\mathbf{B}^\star$ of a basis $\mathbf{B}$:



Decoding radius with $\mathbf{G}^\star$



Decoding radius with $\mathbf{B}^\star$

- Worst-case distance: $\frac{1}{2}\sqrt{\sum \|\mathbf{b}_i^\star\|^2}$ (Approx-CVP)
- Correct decoding of $\mathbf{t} = \mathbf{v} + \mathbf{e}$ where $\mathbf{v} \in \Lambda$ if (BDD)

$$\|\mathbf{e}\| \leq \frac{1}{2} \min \|\mathbf{b}_i^\star\|$$

Profile of a Basis

Good basis

$$\max \|\mathbf{b}_i^*\| \approx \min \|\mathbf{b}_i^*\|$$

Bad basis

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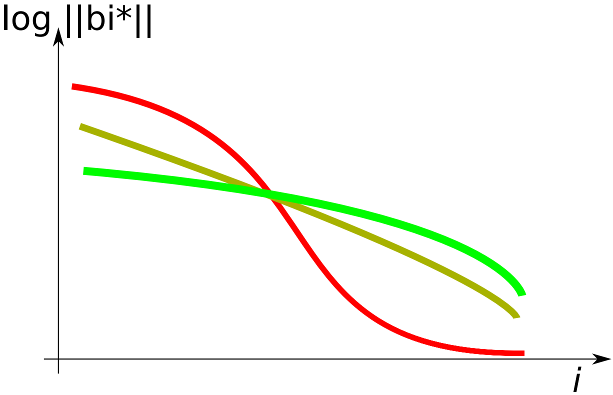

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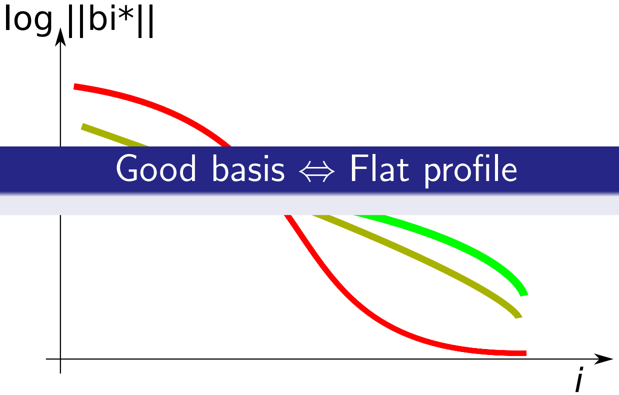

Good basis $\Leftrightarrow$ Flat profile

How Good can a Basis be ?

Theorem (Minkowski)

Let $C \in \mathbb{R}^n$ be a symmetric[a], convex and measurable set.
Let $\Lambda \in \mathbb{R}^n$ be a full dimensional lattice.
If $\mathrm{vol}(C) > 2^n \mathrm{vol}(\Lambda)$ then $\Lambda \cap C$ contains a non-zero point.

[a] $C = -C$

Applied to a Euclidean ball, we get

Corollary

Denoting $\lambda_1(\Lambda)$ the length of a shortest non-zero vector of the n-dimensional lattice Λ, and $\mathfrak{B}_n$ the unit euclidean ball

$$\lambda_1(\Lambda) \leq 2(\mathrm{vol}(\Lambda)/\mathrm{vol}(\mathfrak{B}_n))^{1/n} \qquad \approx \sqrt{2n/\pi e} \cdot \mathrm{vol}(\Lambda)^{1/n}$$

How Good can a Basis be ?

This shows that there exists a basis where $\|\mathbf{b}_1\| = \mathrm{vol}(\Lambda)^{1/n} \cdot O(\sqrt{n})$.
However if λ_1 gets too small, no good basis can exists !

Example (Layered lattice)

Consider the lattice generated by the following basis:

$$\begin{bmatrix} \epsilon & 0 \\ 0 & 1/\epsilon \end{bmatrix}$$

We have $\mathrm{vol}(\Lambda) = 1$, but all its basis $\mathbf{B}$ must have a vector of length $\geq 1/\epsilon$.

Solution:

1. Ignore such corner cases and focus on random lattices

OR 2. State quality of a basis wrt the property of the lattice

Dimension 2

The Wristwatch Lemma

Theorem (Wristwatch lemma)

Let Λ be a 2-dimensional lattice. Then there exists a basis $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2)$ such that

- $\mathbf{b}_1$ *is a shortest vector of* Λ.
- $|\langle \mathbf{b}_1, \mathbf{b}_2 \rangle| \leq \frac{1}{2} \|\mathbf{b}_1\|^2$.

$$\Rightarrow \|\mathbf{b}_2^\star\| \geq \sqrt{3/4} \cdot \|\mathbf{b}_1\|$$

$$\Rightarrow \mathrm{vol}(\Lambda) \geq \sqrt{3/4} \cdot \|\mathbf{b}_1\|^2$$

$$\alpha^2 + \beta^2 \geq 1$$

$$|\alpha| \leq 1/2$$

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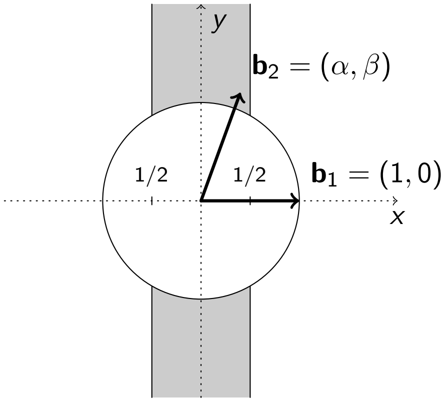

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Proof by Algorithm (Lagrange)

Require: A basis $(\mathbf{b}_1, \mathbf{b}_2)$ of a lattice Λ.
Ensure: A basis $(\mathbf{b}_1, \mathbf{b}_2)$ as in the Wristwatch lemma.

repeat
  swap $\mathbf{b}_1 \leftrightarrow \mathbf{b}_2$
  $k \leftarrow \lceil \langle \mathbf{b}_1, \mathbf{b}_2 \rangle / \|\mathbf{b}_1\|^2 \rfloor$
  $\mathbf{b}_2 \leftarrow \mathbf{b}_2 - k\mathbf{b}_1$
until $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$

Proof:

- Termination: $\|\mathbf{b}_1\|$ strictly decreasing within a discrete set
- $\mathbf{b}_1, \mathbf{b}_2$ is a basis: each step is a transformation in $GL_2(\mathbb{Z})$
- $|\langle \mathbf{b}_1, \mathbf{b}_2 \rangle| \leq \frac{1}{2}\|\mathbf{b}_1\|^2$: because we made sure of that
- $\mathbf{b}_1$ is the shortest lattice vector: left as exercise

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Lemma

The Lagrange reduction algorithm terminates after $O\left(\log \frac{\|\mathbf{b}_1\|}{\sqrt{\det \Lambda}}\right)$ *iterations.*

Proof idea: $\mathbf{b}_1$ decrease by a factor 2 at each step, expect maybe for the last 25 steps.

The Algorithm is (very) Fast

Require: A basis $(\mathbf{b}_1, \mathbf{b}_2)$ of a lattice Λ.
Ensure: A basis $(\mathbf{b}_1, \mathbf{b}_2)$ as in the Wristwatch lemma.

```
repeat
    swap b1 ↔ b2
    k ← ⌈⟨b1, b2⟩/‖b1‖^2⌋
    b2 ← b2 − k b1
until ‖b1‖ ≤ ‖b2‖
```

repeat
- swap $\mathbf{b}_1 \leftrightarrow \mathbf{b}_2$
- $k \leftarrow \lceil \langle \mathbf{b}_1, \mathbf{b}_2 \rangle / \|\mathbf{b}_1\|^2 \rfloor$
- $\mathbf{b}_2 \leftarrow \mathbf{b}_2 - k\mathbf{b}_1$

until $\|\mathbf{b}_1\| \leq \|\mathbf{b}_2\|$

Lemma

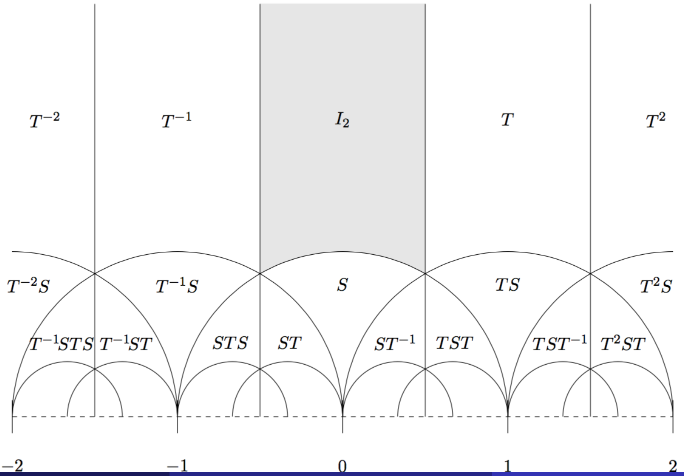
The Lagrange reduction algorithm terminates after $O\left(\log \frac{\|\mathbf{b}_1\|}{\sqrt{\det \Lambda}}\right)$ *iterations.*

Proof idea: $\mathbf{b}_1$ decrease by a factor 2 at each step, expect maybe for the last 25 steps.

Tesselation

Figure: Action of $GL_2(\mathbb{Z})$ over $GL_2(\mathbb{R})$ (modulo rotations and scaling).

$$T : \mathbf{b}_2 \leftarrow \mathbf{b}_2 + \mathbf{b}_1; \qquad S : \mathbf{b}_1 \leftrightarrow \mathbf{b}_2$$



LLL

Projected Sublattices

For a given basis $\mathbf{B}$ of the lattice L

- Define π_i: the projection orthogonally to $\mathbf{b}_1, \ldots, \mathbf{b}_{i-1}$.
- Define $\mathbf{B}_{[i:j]} = (\pi_i(\mathbf{b}_i), \ldots \pi_i(\mathbf{b}_j))$
- $L_{[i:j]}$ is the *projected sublattice* of L spanned by $\mathbf{B}_{[i:j]}$

Identities:

- For $i \leq i' \leq j' \leq j$: $(\mathbf{B}_{[i:j]})_{[i':j']} = \mathbf{B}_{[i':j']}$
- $(\mathbf{B}_{[i:j]})^\star = (\mathbf{b}_i^\star, \ldots, \mathbf{b}_j^\star)$

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Everybody loves commutative diagrams

Relations between the projected sublattices:

$$
\begin{array}{ccccccccccc}
L = & L_{[1:n]} & \supset & L_{[1:n-1]} & \supset & \dots & \supset & L_{[1:2]} & \supset & L_{[1:1]} \\
& \downarrow & & \downarrow & & & & \downarrow & & \\
& L_{[2:n]} & \supset & L_{[2:n-1]} & \supset & \dots & \supset & L_{[2:2]} & & \\
& \downarrow & & \downarrow & & & & & & \\
& \vdots & & \vdots & & \cdot & & & & \\
& \downarrow & & \downarrow & & & & & & \\
& L_{[n-1:n]} & \supset & L_{[n-1:n-1]} & & & & & & \\
& \downarrow & & & & & & & & \\
& L_{[n:n]} & & & & & & & &
\end{array}
$$

$\downarrow$: Projection orthogonally to $\mathbf{b}_i$ for some i.

LLL-reduced basis

Definition (ϵ-Lovasz reduced)

Let $\epsilon \geq 0$. A 2-dimensional basis $\mathbf{B} = (\mathbf{b}_1, \mathbf{b}_2)$ is said ϵ-Lovasz reduced if

$$\|\mathbf{b}_2^\star\| \geq (\sqrt{3/4} + \epsilon) \cdot \|\mathbf{b}_1\|.$$

Definition (ϵ-LLL reduced (Lenstra-Lenstra-Lovasz 1982))

An n-dim basis $\mathbf{B}$ is said ϵ-LLL reduced if $\mathbf{B}_{[i:i+1]}$ is ϵ-Lovasz reduced for all $i \in \{1, \ldots n-1\}$. Equivalently, for all i,

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LLL-reduced : " Gram-Schmidt norms do not deacrease too fast. "

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Properties of LLL-reduced basis

Theorem

Let $B = (\mathbf{b}_1, \ldots, \mathbf{b}_n)$ *be an* ϵ*-LLL reduced basis. Set* $\Lambda = \mathcal{L}(B)$ *and* $\alpha = \sqrt{4/3} + \epsilon$*. Then, we have*

1. $\|\mathbf{b}_1\| \leq \alpha^{\frac{n-1}{2}} \cdot \det(\Lambda)^{1/n}$ — *Root Hermite-factor Bound*
2. $\|\mathbf{b}_1\| \leq \alpha^{n-1} \cdot \lambda_1(\Lambda)$ — *Approximation factor Bound*
3. $\|\mathbf{b}_i^\star\| \leq \alpha^{n-i} \cdot \lambda_i(\Lambda)$.
4. $\|\mathbf{b}_i\| \leq \alpha^{i-1} \cdot \|\mathbf{b}_i^\star\| \leq \alpha^{n-1} \cdot \lambda_i(\Lambda)$
5. $\prod_{i=1}^n \|\mathbf{b}_i\| \leq \alpha^{\frac{n(n-1)}{2}} \cdot \det(\Lambda)$.

LLL Algorithm

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The LLL algorithm

Require: A basis $\mathbf{B}$ of a lattice Λ.
Ensure: An ϵ-LLL reduced basis $\mathbf{B}$ of the same lattice.
  while $\exists i, \mathbf{B}_{[i:i+1]}$ is not ϵ-Lovasz reduced **do**
    Lagrange reduce it...
  end while

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LLL Efficiency

Lemma (Efficiency of LLL)

For a fixed $\epsilon > 0$, and for integral inputs $\mathbf{B} \in \mathbb{Z}^{n \times n}$, the LLL algorithm runs in polynomial time.

Proof:

- Define a potential

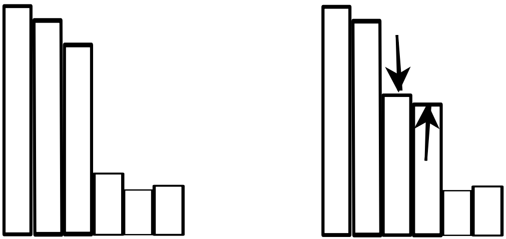
$$P(\mathbf{B}) = \sum (n-i) \log(\|\mathbf{b}_i^\star\|) = \sum \log(\mathrm{vol}(L_{[1:i]}))$$

.

- Each loop of the LLL algorithm decreases $P(\mathbf{B})$ by $\log(1+\epsilon)$
- Note that for integral $\mathbf{B} \in \mathbb{Z}^{n \times n}$, $P(\mathbf{B}) \geq 0$.
- Concludes

$$\#loops \leq \frac{P(\mathbf{B}_{init})}{\log(1+\epsilon)} \leq \frac{n^2 \cdot log(\|\mathbf{B}_{init}\|_\infty)}{\log(1+\epsilon)}.$$

LLL : a 2-legged ant



- Columns height : $\log(\|\mathbf{b}_i^\star\|)$
- Volume invariant: mass of sand is preserved
- Lagrange reduction : almost level 2 consecutive columns
- Potential decreases : the mass of sand moves toward the right.

Remarks

Note the integrality condition

By scaling, can also wrok over $\mathbb{Q}$. However, even assuming perfect computation over $\mathbb{R}$, the above proof fails for real inputs.

The above proof is incomplete

$\#loops = poly$. Ok. But how large are the intermediate values ?

$\epsilon = 0$

The algorithm still terminates, but not poly-time.

Output quality

On most inputs, LLL behave much better in practice than in theory.

- Theory: $\alpha = \sqrt{4/3} + \epsilon \approx 1.155$
- Practice: $\alpha \approx 1.022$

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BKZ: a b-legged ant

Bigger Local Improvement

- Focus on a block $[i:j]$, of dimension $b = j - i - 1$
- Find the shortest vector v of the projected sublattice $L_{[i:j]}$

"a puzzle of it
"the right combination."

- Construct a unimodular matrix $\mathbf{U}$ such that $\mathbf{T}_{[i:j]} \cdot \mathbf{U} = [\mathbf{v}, *, *, \ldots]$. Apply $\mathbf{U}$ (locally).
- The new $\mathbf{b}_i^* = v$ got shorter!
- The other $\mathbf{b}_{i+1}^*, \ldots, \mathbf{b}_j^*$ will change as well

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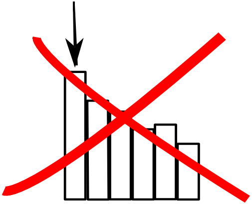

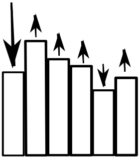

b: Blocksize

Run the local improvements for consecutive blocks:

$[1 : b], [2 : b+1], [3 : b+2], \ldots, [n-b : n], [n-b+1 : n], \ldots \; [n-1 : n]$

This is called a tour.

Repeat tours until satisfication (or convergence).

BKZ-b

b: Blocksize

Run the local improvements for consecutive blocks:

$[1:b], [2:b+1], [3:b+2], \ldots, [n-b:n], [n-b+1:n], \ldots\ [n-1:n]$

This is called a tour.

Repeat tours until satisfication (or convergence).

BKZ prediction

At the end of BKZ, $\mathbf{b}_i^*$ is the shortest vector of $L_{[i:i+b]}$.

Worse case bound (Minkowski)

$$\|\mathbf{b}_i^\star\| \lessapprox \sqrt{\frac{2n}{\pi e}} \cdot \mathrm{vol}(L_{[i:i+b]})^{1/n} = \sqrt{\frac{2n}{\pi e}} \cdot \prod_{k=i}^{i+b} \|\mathbf{b}_k^\star\|^{1/n}$$

Average case (for $b \geq 50$)

$$\|\mathbf{b}_i^\star\| \approx \sqrt{\frac{n}{2\pi e}} \cdot \prod_{k=i}^{i+b} \|\mathbf{b}_k^\star\|^{1/n}$$

Take logs, get a linear recurrence $\Rightarrow$ shape is linear, slope is a known function of blocksize b. Known as the Geometric Series Assumption (GSA).

BKZ time vs. approximation

- Solving SVP-b cost

$$2^{\tilde{\Theta}(b)}$$

.

- BKZ can run for very long. But stopping after poly(n, b) steps is sufficient
- The total cost of BKZ is therefore

$$n^{O(1)} \cdot 2^{\tilde{\Theta}(b)}$$

- Achieve an approximation factor:

$$\|b_1\| \approx (b/2\pi e)^{n/2b} \cdot \mathrm{vol}(L)$$

Enumeration

Enumeration in 1 slide

Goal: enumerate all the lattice points in a ball of radius r : $r\mathfrak{B} \cap L$.
Remember the chain of lattices (by projections)

$$L = L_{[1:n]} \xrightarrow{\pi_1} L_{[2:n]} \xrightarrow{\pi_2} L_{[3:n]} \to \ldots \xrightarrow{\pi_{n-1}} L_{[n:n]} \xrightarrow{\pi_n} \{0\}$$

Projection only decrease length: $\pi_i(r\mathfrak{B} \cap L_{[i:n]}) \subset r\mathfrak{B} \cap L_{[i+1:n]}$.

Enumeration algorithm

Compute sets $S_i = B \cap L_{[i+1:n]}$ using the recursion

$$S_n = \{0\}; \qquad S_{i-1} = \pi_i^{-1}(S_i) \cap r\mathfrak{B}$$

Cost of Enumeration

The cost of the algorithm is essentially proportional to $\sum |S_i|$.

$$|S_i| \approx r^{n-i} \cdot \mathrm{vol}(\mathfrak{B}_{n-i}) / \mathrm{vol}(L_{[i:n]}).$$

Depends projected sublattice volumes: SVP faster if the basis is well reduced !

- LLL reduced : $2^{\Theta(n^2)}$
- BKZ-b where $b = n - O(\log n)$: $n^{n/2e+o(n)}$ (Michael ?)

Optimal complexity is reached using a mixed recursion:

$$\text{BKZ-}b_1 \looparrowright \text{SVP-}b_1 \looparrowright \text{BKZ-}b_2 \looparrowright \cdots \looparrowright \text{SVP-60} \looparrowright \text{LLL}$$