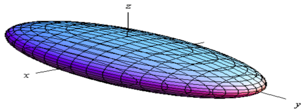
Structured variants of LWE
(and SIS and NTRU)



Wouter Castryck
(contains joint work with Carl Bootland, Ilia Iliashenko, Alan Szepieniec, Frederik Vercauteren)

KU LEUVEN

UNIVERSITEIT GENT

KU Leuven
Dept. of Mathematics & COSIC

Ghent University

1. Learning With Errors [Reg05]

The LWE problem: solve a linear system

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} \approx \begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,n-1} \\ a_{20} & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{m,n-1} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

over a finite field $\mathbb{F}_p$ for a secret $(s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}_p^n$ where

1. Learning With Errors [Reg05]

The LWE problem: solve a linear system

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} \approx \begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,n-1} \\ a_{20} & a_{21} & \dots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{m,n-1} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

over a finite field $\mathbb{F}_p$ for a secret $(s_0, s_1, \dots, s_{n-1}) \in \mathbb{F}_p^n$ where

- each equation is perturbed by a “small” error, i.e.

$$b_i = a_{i0}s_0 + a_{i1}s_1 + \cdots + a_{i,n-1}s_{n-1} + e_i,$$

1. Learning With Errors [Reg05]

The LWE problem: solve a linear system

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} \approx \begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,n-1} \\ a_{20} & a_{21} & \dots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{m,n-1} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

over a finite field $\mathbb{F}_p$ for a secret $(s_0, s_1, \dots, s_{n-1}) \in \mathbb{F}_p^n$ where

- each equation is perturbed by a “small” error, i.e.

$$b_i = a_{i0}s_0 + a_{i1}s_1 + \cdots + a_{i,n-1}s_{n-1} + e_i,$$

- the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random,

1. Learning With Errors [Reg05]

The LWE problem: solve a linear system

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} \approx \begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,n-1} \\ a_{20} & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{m,n-1} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

over a finite field $\mathbb{F}_p$ for a secret $(s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}_p^n$ where

- each equation is perturbed by a "small" error, i.e.

$$b_i = a_{i0}s_0 + a_{i1}s_1 + \cdots + a_{i,n-1}s_{n-1} + e_i,$$

- the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random,
- $m > n$.

1. Learning With Errors [Reg05]

The LWE problem: solve a linear system

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} = \begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,n-1} \\ a_{20} & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{m,n-1} \end{pmatrix} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{m-1} \end{pmatrix}$$

over a finite field $\mathbb{F}_p$ for a secret $(s_0, s_1, \ldots, s_{n-1}) \in \mathbb{F}_p^n$ where

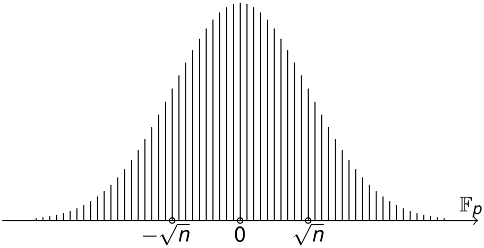
- each equation is perturbed by a “small” error, i.e.

$$b_i = a_{i0}s_0 + a_{i1}s_1 + \cdots + a_{i,n-1}s_{n-1} + e_i,$$

- the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random,
- $m > n$.

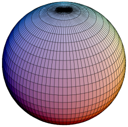
1. Learning With Errors [Reg05]

The errors e_i are sampled independently from a discretized Gaussian with standard deviation $\sigma \gtrsim \sqrt{n}$:



When viewed jointly, the error vector

$$\begin{pmatrix} e_0 \\ \vdots \\ e_{m-1} \end{pmatrix}$$



is sampled from a spherical Gaussian.

1. Learning With Errors [Reg05]

Known attacks for $q = \mathrm{poly}(n)$:

- Trial and error:
  $2^{O(n \log n)}$ time and $O(n)$ samples.
- A. Blum, A. Kalai, H. Wasserman '03:
  $2^{O(n)}$ time and $2^{O(n)}$ samples.
- S. Arora, R. Ge '11:
  $2^{O(\sigma^2 \log n)}$ time and $2^{O(\sigma^2 \log n)}$ samples.

1. Learning With Errors [Reg05]

Known attacks for $q = \text{poly}(n)$:

- Trial and error:
  $2^{O(n \log n)}$ time and $O(n)$ samples.
- A. Blum, A. Kalai, H. Wasserman '03:
  $2^{O(n)}$ time and $2^{O(n)}$ samples.
- S. Arora, R. Ge '11:
  $2^{O(\sigma^2 \log n)}$ time and $2^{O(\sigma^2 \log n)}$ samples.
  Idea: if all errors (almost) certainly lie in $\{-T, \ldots, T\}$, then

$$\prod_{j=-T}^{T} \left(a_{i,0}s_0 + a_{i,1}s_1 + \cdots + a_{i,n-1}s_{n-1} - b_i - j\right) = 0.$$

1. Learning With Errors [Reg05]

Known attacks for $q = \text{poly}(n)$:

- Trial and error:
  $2^{O(n \log n)}$ time and $O(n)$ samples.
- A. Blum, A. Kalai, H. Wasserman '03:
  $2^{O(n)}$ time and $2^{O(n)}$ samples.
- S. Arora, R. Ge '11:
  $2^{O(\sigma^2 \log n)}$ time and $2^{O(\sigma^2 \log n)}$ samples.
  Idea: if all errors (almost) certainly lie in $\{-T, \ldots, T\}$, then

$$\prod_{j=-T}^{T} \left(a_{i,0}s_0 + a_{i,1}s_1 + \cdots + a_{i,n-1}s_{n-1} - b_i - j\right) = 0.$$

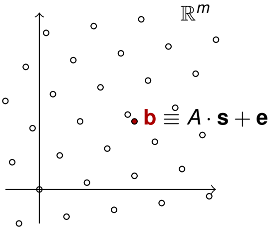
  View as linear system of equations in $\approx n^{2T}$ monomials.

1. Learning With Errors [Reg05]

LWE is tightly related to classical lattice problems.

- Can be thought of as an instance of BDD inside the lattice

$$\begin{array}{c} (p\mathbb{Z})^m \\ \cap \\ \{\mathbf{w} \in \mathbb{Z}^m \mid \exists \mathbf{s} \in \mathbb{Z}^n : \mathbf{w} \equiv A \cdot \mathbf{s} \bmod p\} \\ \cap \\ \mathbb{Z}^m. \end{array}$$

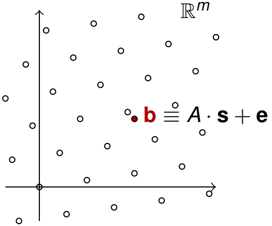


1. Learning With Errors [Reg05]

LWE is tightly related to classical lattice problems.

- Can be thought of as an instance of BDD inside the lattice

$$\begin{array}{c} (p\mathbb{Z})^m \\ \cap \\ \{\mathbf{w} \in \mathbb{Z}^m \mid \exists \mathbf{s} \in \mathbb{Z}^n : \mathbf{w} \equiv A{\cdot}\mathbf{s} \bmod p\} \\ \cap \\ \mathbb{Z}^m. \end{array}$$



- Proven to be at least as hard as quantum SIVP.

1. Learning With Errors (LWE)

Features:

- hardness reduction from famous lattice problems,
- versatile building block for cryptography, enabling exciting applications (post-quantum crypto, FHE, . . .)

1. Learning With Errors (LWE)

Features:

- hardness reduction from famous lattice problems,
- versatile building block for cryptography, enabling exciting applications (post-quantum crypto, FHE, . . .)

Drawback: key size.

- To hide the secret one needs an entire linear system:

$$\underset{\substack{\uparrow \\ m \log p}}{\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix}} \approx \underset{\substack{\uparrow \\ mn \log p}}{\begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,n-1} \\ a_{20} & a_{21} & \cdots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \cdots & a_{m,n-1} \end{pmatrix}} \cdot \underset{\substack{\uparrow \\ n \log p}}{\begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}}.$$

2. Ring-based LWE?

Idea:

- Identify key space

$$\mathbb{F}_p^n \qquad \text{with} \qquad \frac{\mathbb{Z}[x]}{(p, f(x))}$$

for some monic deg n polynomial $f(x) \in \mathbb{Z}[x]$, by viewing

$$(s_0, s_1, \ldots, s_{n-1}) \quad \text{as} \quad s_0 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}.$$

2. Ring-based LWE?

Idea:

- Identify key space

$$\mathbb{F}_p^n \qquad \text{with} \qquad \frac{\mathbb{Z}[x]}{(p, f(x))}$$

for some monic deg n polynomial $f(x) \in \mathbb{Z}[x]$, by viewing

$$(s_0, s_1, \ldots, s_{n-1}) \quad \text{as} \quad s_0 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}.$$

- Replace every block of n eqns by a block of the form

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \approx A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

with $A_{\mathbf{a}}$ the matrix of multiplication by some random $\mathbf{a}(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$.

2. Ring-based LWE?

Idea:

- Identify key space

$$\mathbb{F}_p^n \qquad \text{with} \qquad \frac{\mathbb{Z}[x]}{(p, f(x))}$$

for some monic deg n polynomial $f(x) \in \mathbb{Z}[x]$, by viewing

$$(s_0, s_1, \ldots, s_{n-1}) \quad \text{as} \quad s_0 + s_1 x + s_2 x^2 + \cdots + s_{n-1} x^{n-1}.$$

- Replace every block of n eqns by a block of the form

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} \approx A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix}$$

with $A_{\mathbf{a}}$ the matrix of multiplication by some random $\mathbf{a}(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$.

- Store $\mathbf{a}(x)$ rather than $A_{\mathbf{a}}$: saves factor n.

2. Ring-based LWE?

Example:

- if $f(x) = x^n - 1$, then $A_{\mathbf{a}}$ is a circulant matrix

$$\begin{pmatrix} a_0 & a_{n-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & \dots & a_3 & a_2 \\ a_2 & a_1 & \dots & a_4 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{pmatrix}$$

of which it suffices to store the first column.

2. Ring-based LWE?

Example:

- if $f(x) = x^n - 1$, then $A_{\mathbf{a}}$ is a circulant matrix

$$\begin{pmatrix} a_0 & a_{n-1} & \dots & a_2 & a_1 \\ a_1 & a_0 & \dots & a_3 & a_2 \\ a_2 & a_1 & \dots & a_4 & a_3 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{n-1} & a_{n-2} & \dots & a_1 & a_0 \end{pmatrix}$$

of which it suffices to store the first column.

- Bad example, because of ...

3. Ring-based LWE?

Potential threat:

smallness preserving homomorphisms to smaller rings.

- Suppose e.g. that $f(1) \equiv 0 \bmod p$, then

$$R_p := \frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

  is a well-defined ring homomorphism.

3. Ring-based LWE?

Potential threat:

smallness preserving homomorphisms to smaller rings.

- Suppose e.g. that $f(1) \equiv 0 \bmod p$, then

$$R_p := \frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

  is a well-defined ring homomorphism.
- Our ring-based LWE samples

$$\mathbf{b}(x) = \mathbf{a}(x) \cdot \mathbf{s}(x) + \mathbf{e}(x)$$

  evaluate to

$$\mathbf{b}(1) = \mathbf{a}(1) \cdot \mathbf{s}(1) + \mathbf{e}(1).$$

3. Ring-based LWE?

Potential threat:

smallness preserving homomorphisms to smaller rings.

- Suppose e.g. that $f(1) \equiv 0 \bmod p$, then

$$R_p := \frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

  is a well-defined ring homomorphism.
- Our ring-based LWE samples

$$\mathbf{b}(x) = \mathbf{a}(x) \cdot \mathbf{s}(x) + \mathbf{e}(x)$$

  evaluate to

$$\mathbf{b}(1) = \mathbf{a}(1) \cdot \mathbf{s}(1) + \mathbf{e}(1).$$

- For each guess for $\mathbf{s}(1) \in \mathbb{F}_p$, analyze distribution of $\mathbf{e}(1)$.

3. Ring-based LWE?

Potential threat:

smallness preserving homomorphisms to smaller rings.

- Suppose e.g. that $f(1) \equiv 0 \bmod p$, then

$$R_p := \frac{\mathbb{Z}[x]}{(p, f(x))} \to \mathbb{F}_p : \mathbf{r}(x) \mapsto \mathbf{r}(1) = r_0 + r_1 + \cdots + r_{n-1},$$

  is a well-defined ring homomorphism.
- Our ring-based LWE samples

$$\mathbf{b}(x) = \mathbf{a}(x) \cdot \mathbf{s}(x) + \mathbf{e}(x)$$

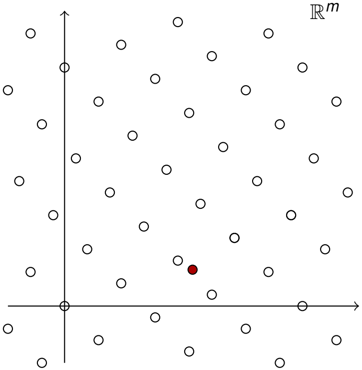
  evaluate to

$$\mathbf{b}(1) = \mathbf{a}(1) \cdot \mathbf{s}(1) + \mathbf{e}(1).$$

- For each guess for $\mathbf{s}(1) \in \mathbb{F}_p$, analyze distribution of $\mathbf{e}(1)$.
- Non-uniformity might reveal $\mathbf{s}(1)$.

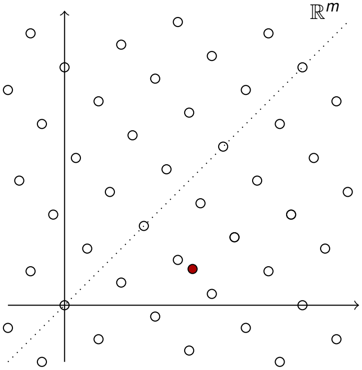
3. Ring-based LWE?

A lattice point of view:



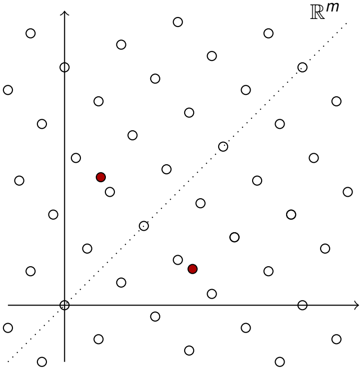
3. Ring-based LWE?

A lattice point of view:



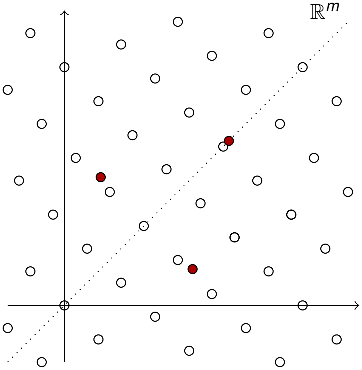
3. Ring-based LWE?

A lattice point of view:



3. Ring-based LWE?

A lattice point of view:



3. Ring-based LWE?

Safety measure: restrict to irreducible $f(x) \in \mathbb{Z}[x]$.

- Rules out examples like $x^n - 1$.
- Resulting problem is often called Poly-LWE [SSTX09].
- Notice: our 'parent ring'

$$R = \frac{\mathbb{Z}[x]}{(f(x))}$$

is an order in the number field $K = \mathbb{Q}[x]/(f(x))$.

3. Ring-based LWE?

Safety measure: restrict to irreducible $f(x) \in \mathbb{Z}[x]$.

- Rules out examples like $x^n - 1$.
- Resulting problem is often called Poly-LWE [SSTX09].
- Notice: our 'parent ring'

$$R = \frac{\mathbb{Z}[x]}{(f(x))}$$

is an order in the number field $K = \mathbb{Q}[x]/(f(x))$.

Does this really solve our problem?

- No! E.g., $f(x) = x^n + (p-1)$ suffers from same problem.
- Possible to make examples where $K/\mathbb{Q}$ is Galois [EHL14].
  $\rightsquigarrow$ $\mathbf{s}(1)$ is enough to reconstruct $\mathbf{s}$ completely!

3. Ring-based LWE?

Safety measure: restrict to irreducible $f(x) \in \mathbb{Z}[x]$.

- Rules out examples like $x^n - 1$.
- Resulting problem is often called Poly-LWE [SSTX09].
- Notice: our 'parent ring'

$$R = \frac{\mathbb{Z}[x]}{(f(x))}$$

is an order in the number field $K = \mathbb{Q}[x]/(f(x))$.

Does this really solve our problem?

- No! E.g., $f(x) = x^n + (p - 1)$ suffers from same problem.
- Possible to make examples where $K/\mathbb{Q}$ is Galois [EHL14].
  $\rightsquigarrow$ $\mathbf{s}(1)$ is enough to reconstruct $\mathbf{s}$ completely!

Ring-LWE: choose more 'canonical' error distribution [LPR12].

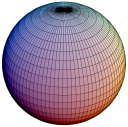
4. Ring-LWE [LPR12]

Direct ring-based analogue of LWE-sample would read

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

with the e_i sampled independently from

$$N(0, \sigma)$$



for some fixed small $\sigma = \sigma(n)$.

This is not Ring-LWE!

4. Ring-LWE [LPR12]

So what is Ring-LWE? Samples look like

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

4. Ring-LWE [LPR12]

So what is Ring-LWE? Samples look like

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

where

- B is the canonical embedding matrix,
- $A_{f'(x)}$ compensates for the fact that one actually picks secrets from the dual.

4. Ring-LWE [LPR12]

So what is Ring-LWE? Samples look like

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

where

- B is the canonical embedding matrix,
- $A_{f'(x)}$ compensates for the fact that one actually picks secrets from the dual.

At least as hard as quantum Ideal-SIVP.

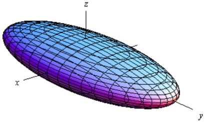
4. Ring-LWE [LPR12]

So what is Ring-LWE? Samples look like

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

where

- B is the canonical embedding matrix,
- $A_{f'(x)}$ compensates for the fact that one actually picks secrets from the dual.



At least as hard as quantum Ideal-SIVP.

Note:

- factor $A_{f'(x)} \cdot B^{-1}$ might skew the error distribution,

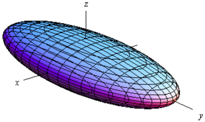
4. Ring-LWE [LPR12]

So what is Ring-LWE? Samples look like

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

where

- B is the canonical embedding matrix,
- $A_{f'(x)}$ compensates for the fact that one actually picks secrets from the dual.



At least as hard as quantum Ideal-SIVP.

Note:

- factor $A_{f'(x)} \cdot B^{-1}$ might skew the error distribution,
- but also scales it!

4. Ring-LWE [LPR12]

...but also scales it!

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

Indeed, one has

- $\det A_{f'(x)} = \Delta$ with

$$\Delta = |\mathrm{disc}\, f(x)|\,, \qquad \leftarrow \text{typically huge}$$

4. Ring-LWE [LPR12]

...but also scales it!

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

Indeed, one has

- $\det A_{f'(x)} = \Delta$ with

$$\Delta = \left|\operatorname{disc} f(x)\right|, \qquad \leftarrow \text{typically huge}$$

- $\det B^{-1} = 1/\sqrt{\Delta}$.

4. Ring-LWE [LPR12]

…but also scales it!

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + {\color{red}A_{f'(x)}} \cdot {\color{blue}B^{-1}} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

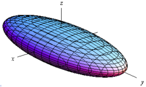
Indeed, one has

- det ${\color{red}A_{f'(x)}} = \Delta$ with

$$\Delta = |\mathrm{disc}\, f(x)|\,, \qquad \leftarrow \text{typically huge}$$

- det ${\color{blue}B^{-1}} = 1/\sqrt{\Delta}$.

So "on average", each e_i is scaled up by $\sqrt{\Delta}^{1/n}$ …



- … but remember: skewness.

4. Ring-LWE [LPR12]

Main example: 2-power cyclotomics $f(x) = x^n + 1$ with $n = 2^k$.

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + A_{f'(x)} \cdot B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

- $f'(x) = nx^{n-1} = n \times$ unit, so

$$A_{f'(x)} = n \times \text{orthogonal matrix},$$

- all singular values of B are $\sqrt{n}$, so

$$B^{-1} = \frac{1}{\sqrt{n}} \times \text{orthogonal matrix},$$

4. Ring-LWE [LPR12]

Main example: 2-power cyclotomics $f(x) = x^n + 1$ with $n = 2^k$.

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \sqrt{n} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}$$

- $f'(x) = nx^{n-1} = n \times$ unit, so

$$A_{f'(x)} = n \times \text{orthogonal matrix},$$

- all singular values of B are $\sqrt{n}$, so

$$B^{-1} = \frac{1}{\sqrt{n}} \times \text{orthogonal matrix},$$

Therefore Ring-LWE = Poly-LWE in this case.

5. A wrong instantiation

Recall: successful attack on Ring-LWE

$$\Downarrow$$

quantum solution to SIVP in ideal lattices.

5. A wrong instantiation

Recall: successful attack on Ring-LWE

$$\Downarrow$$

quantum solution to SIVP in ideal lattices.

[ELOS15] announced successful evaluation-at-1 attack

$\rightsquigarrow$ but for convenience picked non-dual secrets:

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \cancel{A_{f'(x)}} \; B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}.$$

5. A wrong instantiation

Recall: successful attack on Ring-LWE

$$\Downarrow$$

quantum solution to SIVP in ideal lattices.

[ELOS15] announced successful evaluation-at-1 attack

$\rightsquigarrow$ but for convenience picked non-dual secrets:

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \cancel{A_{f'(x)}} \; B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}.$$

But:

- $\det B^{-1} = 1/\sqrt{\Delta}$, so the errors get squeezed.

5. A wrong instantiation

Recall: successful attack on Ring-LWE

$$\Downarrow$$

quantum solution to SIVP in ideal lattices.

[ELOS15] announced successful evaluation-at-1 attack

$\rightsquigarrow$ but for convenience picked non-dual secrets:

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \cancel{A_{f'(x)}}\ B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}.$$

But:

- $\det B^{-1} = 1/\sqrt{\Delta}$, so the errors get squeezed.
- To compensate, they scale up the errors by a factor $\sqrt{\Delta}^{1/n}$.

5. A wrong instantiation

Issue:

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \sqrt{\Delta}^{1/n} B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}.$$

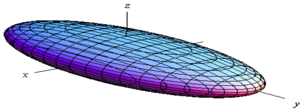
- The factor $\sqrt{\Delta}^{1/n}$ compensates for B^{-1} only "on average".

5. A wrong instantiation

Issue:

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \end{pmatrix} = A_{\mathbf{a}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \sqrt{\Delta}^{1/n} B^{-1} \cdot \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{n-1} \end{pmatrix}.$$

- The factor $\sqrt{\Delta}^{1/n}$ compensates for B^{-1} only "on average".
- In some coordinates B^{-1} could scale down much more.

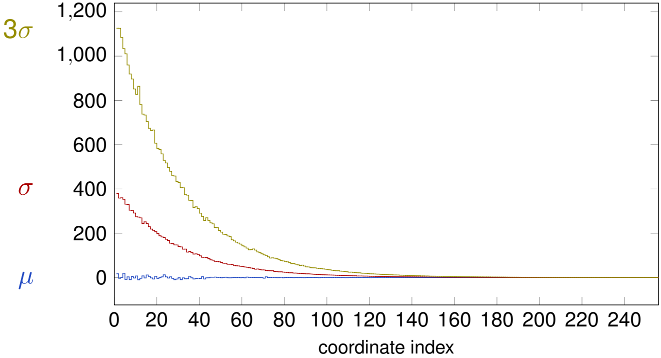


Compensation factor is insufficient

$\rightsquigarrow$ merely rounding yields exact equations in the secret!

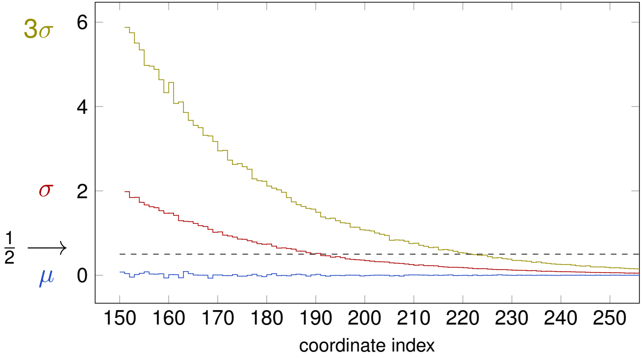
5. A wrong instantiation

- Concrete example: $f(x) = x^{256} + 8190$, $p = 8191$.
- Standard deviations even form a geometric series! Error distribution in each coordinate (experimental):



5. A wrong instantiation

- Concrete example: $f(x) = x^{256} + 8190$, $p = 8191$.
- Standard deviations even form a geometric series!
  Error distribution in each coordinate (experimental):



6. Module-LWE [LS15]

Recall: LWE is about solving a noisy system of linear equations

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} = \underset{\substack{\| \\ \displaystyle A}}{\begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,n-1} \\ a_{20} & a_{21} & \dots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{m,n-1} \end{pmatrix}} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{m-1} \end{pmatrix}$$

in $\mathbb{F}_p^n$.

6. Module-LWE [LS15]

Recall: LWE is about solving a noisy system of linear equations

$$\begin{pmatrix} b_0 \\ b_1 \\ \vdots \\ b_{m-1} \end{pmatrix} = \underbrace{\begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,n-1} \\ a_{20} & a_{21} & \dots & a_{2,n-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m0} & a_{m1} & \dots & a_{m,n-1} \end{pmatrix}}_{\parallel \atop A} \cdot \begin{pmatrix} s_0 \\ s_1 \\ \vdots \\ s_{n-1} \end{pmatrix} + \begin{pmatrix} e_0 \\ e_1 \\ \vdots \\ e_{m-1} \end{pmatrix}$$

in $\mathbb{F}_p^n$.

In Ring-LWE we replace A by a matrix of multiplication $A_{\mathbf{a}}$ with

$$\mathbf{a} \in R_p = \frac{\mathbb{Z}[x]}{(p, f(x))} \cong \mathbb{F}_p^n.$$

6. Module-LWE [LS15]

Let $n = \ell \cdot \ell'$. Module-LWE is about solving a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \cdots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \cdots & \mathbf{a}_{k,\ell-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

in R_p^ℓ,

6. Module-LWE [LS15]

Let $n = \ell \cdot \ell'$. Module-LWE is about solving a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \cdots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \cdots & \mathbf{a}_{k,\ell-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

in R_p^{ℓ}, where

$$R_p = \frac{\mathbb{Z}[x]}{(f(x))}, \qquad f(x) \text{ monic irreducible of degree } \ell',$$

and all $\mathbf{e}_i$ are sampled as in Ring-LWE.

6. Module-LWE [LS15]

Let $n = \ell \cdot \ell'$. Module-LWE is about solving a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \dots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \dots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \dots & \mathbf{a}_{k,\ell-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

in R_p^ℓ, where

$$R_p = \frac{\mathbb{Z}[x]}{(f(x))}, \qquad f(x) \text{ monic irreducible of degree } \ell',$$

and all $\mathbf{e}_i$ are sampled as in Ring-LWE.

This fills A blockwise with matrices of multiplication.

6. Module-LWE [LS15]

Let $n = \ell \cdot \ell'$. Module-LWE is about solving a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \cdots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \cdots & \mathbf{a}_{k,\ell-1} \end{pmatrix} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

in R_p^ℓ, where

$$R_p = \frac{\mathbb{Z}[x]}{(f(x))}, \qquad f(x) \text{ monic irreducible of degree } \ell',$$

and all $\mathbf{e}_i$ are sampled as in Ring-LWE.

This fills A blockwise with matrices of multiplication.

At least as hard as quantum Module-SIVP.

7. Variants of NTRU [HPS98], [CG05]

Matrix version of the NTRU problem:

- Consider two $n \times n$ matrices

$$A = (a_{ij}), \qquad B = (b_{ij}) \qquad \in \mathbb{F}_p^{n \times n}$$

with a_{ij}, b_{ij} sampled randomly from a narrow distribution. If $\det B = 0$, start over.

7. Variants of NTRU [HPS98], [CG05]

Matrix version of the NTRU problem:

- Consider two $n \times n$ matrices

$$A = (a_{ij}), \qquad B = (b_{ij}) \qquad \in \mathbb{F}_p^{n \times n}$$

  with a_{ij}, b_{ij} sampled randomly from a narrow distribution. If $\det B = 0$, start over.

- Compute

$$H = AB^{-1} \in \mathbb{F}_p^{n \times n}.$$

7. Variants of NTRU [HPS98], [CG05]

Matrix version of the NTRU problem:

- Consider two $n \times n$ matrices

$$A = (a_{ij}), \qquad B = (b_{ij}) \qquad \in \mathbb{F}_p^{n\times n}$$

  with a_{ij}, b_{ij} sampled randomly from a narrow distribution. If $\det B = 0$, start over.

- Compute

$$H = AB^{-1} \in \mathbb{F}_p^{n\times n}.$$

- Problem: given H, find small $A, B \in \mathbb{F}_p^{n\times n}$ with $H = AB^{-1}$.

7. Variants of NTRU [HPS98], [CG05]

Matrix version of the NTRU problem:

- Consider two $n \times n$ matrices

$$A = (a_{ij}), \qquad B = (b_{ij}) \qquad \in \mathbb{F}_p^{n\times n}$$

  with a_{ij}, b_{ij} sampled randomly from a narrow distribution. If $\det B = 0$, start over.

- Compute

$$H = AB^{-1} \in \mathbb{F}_p^{n\times n}.$$

- Problem: given H, find small $A, B \in \mathbb{F}_p^{n\times n}$ with $H = AB^{-1}$.

Best-known version of the NTRU problem:

- Replace A, B by matrices of multiplication $A_{\mathbf{a}}, B_{\mathbf{b}}$ for small

$$\mathbf{a}, \mathbf{b} \in R_p = \mathbb{Z}[x]/(f(x)), \qquad f(x) \text{ monic irr. of deg } n.$$

7. Variants of NTRU [HPS98], [CG05]

Module version of NTRU:

- Let $n = \ell \times \ell'$. Consider two $\ell \times \ell$ matrices

$$A = (\mathbf{a}_{ij}), \qquad B = (\mathbf{b}_{ij}) \qquad \in \; R_p^{\ell \times \ell}$$

with

$$\mathbf{a}_{ij}, \mathbf{b}_{ij} \in R_p = \mathbb{Z}[x]/(f(x)), \quad f(x) \text{ monic irr. of deg } \ell'$$

sampled randomly from a narrow distribution.
If B not invertible, start over.

7. Variants of NTRU [HPS98], [CG05]

Module version of NTRU:

- Let $n = \ell \times \ell'$. Consider two $\ell \times \ell$ matrices

$$A = (\mathbf{a}_{ij}), \qquad B = (\mathbf{b}_{ij}) \qquad \in \ R_p^{\ell \times \ell}$$

  with

$$\mathbf{a}_{ij}, \mathbf{b}_{ij} \in R_p = \mathbb{Z}[x]/(f(x)), \quad f(x) \text{ monic irr. of deg } \ell'$$

  sampled randomly from a narrow distribution.
  If B not invertible, start over.
- Compute

$$H = AB^{-1} \in R_p^{\ell \times \ell}.$$

7. Variants of NTRU [HPS98], [CG05]

Module version of NTRU:

- Let $n = \ell \times \ell'$. Consider two $\ell \times \ell$ matrices

$$A = (\mathbf{a}_{ij}), \qquad B = (\mathbf{b}_{ij}) \qquad \in \; R_p^{\ell \times \ell}$$

  with

$$\mathbf{a}_{ij}, \mathbf{b}_{ij} \in R_p = \mathbb{Z}[x]/(f(x)), \quad f(x) \text{ monic irr. of deg } \ell'$$

  sampled randomly from a narrow distribution.
  If B not invertible, start over.

- Compute

$$H = AB^{-1} \in R_p^{\ell \times \ell}.$$

- Problem: given H, find small $A, B \in R_p^{\ell \times \ell}$ with $H = AB^{-1}$.

7. Variants of NTRU [HPS98], [CG05]

Remark: if ℓ is small, e.g.,

$$H = \frac{A}{B} = \frac{\begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix}}{\begin{pmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{pmatrix}} \in R_p^{2\times 2}$$

7. Variants of NTRU [HPS98], [CG05]

Remark: if ℓ is small, e.g.,

$$H = \frac{A}{B} = \frac{\begin{pmatrix} \mathbf{a}_{11} & \mathbf{a}_{12} \\ \mathbf{a}_{21} & \mathbf{a}_{22} \end{pmatrix}}{\begin{pmatrix} \mathbf{b}_{11} & \mathbf{b}_{12} \\ \mathbf{b}_{21} & \mathbf{b}_{22} \end{pmatrix}} \in R_p^{2\times 2}$$

then taking determinants yields

$$\det H = \frac{\det A}{\det B} = \frac{\mathbf{a}_{11}\mathbf{a}_{22} - \mathbf{a}_{21}\mathbf{a}_{12}}{\mathbf{b}_{11}\mathbf{b}_{22} - \mathbf{b}_{21}\mathbf{b}_{12}}$$

which is an NTRU-instance in R_p; may suffice to recover A, B.

8. Variants of SIS [Aj96], [LM06], [PR06], [LS15]

The SIS problem is about finding a small solution in $\mathbb{F}_p^m \setminus \{0\}$ to

$$\begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,m-1} \\ a_{20} & a_{21} & \cdots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{n,m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} = 0$$

where $n < m$ and the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random.

8. Variants of SIS [Aj96], [LM06], [PR06], [LS15]

The SIS problem is about finding a small solution in $\mathbb{F}_p^m \setminus \{0\}$ to

$$\begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,m-1} \\ a_{20} & a_{21} & \dots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{n,m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} = 0$$

where $n < m$ and the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random.

One defines:

- Ring-SIS (assume $m = kn$):
  - Find small $\mathbf{x}_i \in R_p = \mathbb{Z}[x]/(f(x))$ with $\deg f(x) = n$ such that

$$\begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_k \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_k \end{pmatrix}^T = 0$$

    with $\mathbf{a}_i \in R_p$ random.

8. Variants of SIS [Aj96], [LM06], [PR06], [LS15]

The SIS problem is about finding a small solution in $\mathbb{F}_p^m \setminus \{0\}$ to

$$\begin{pmatrix} a_{10} & a_{11} & \cdots & a_{1,m-1} \\ a_{20} & a_{21} & \cdots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \cdots & a_{n,m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} = 0$$

where $n < m$ and the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random.

One defines:

- Ring-SIS (assume $m = kn$):
  - Find small $\mathbf{x}_i \in R_p = \mathbb{Z}[x]/(f(x))$ with $\deg f(x) = n$ such that

    $$\begin{pmatrix} \mathbf{a}_1 & \ldots & \mathbf{a}_k \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1 & \ldots & \mathbf{x}_k \end{pmatrix}^T = 0$$

    with $\mathbf{a}_i \in R_p$ random.
- Module-SIS: similar (fill matrix with blocks)

8. Variants of SIS [Aj96], [LM06], [PR06], [LS15]

The SIS problem is about finding a small solution in $\mathbb{F}_p^m \setminus \{0\}$ to

$$\begin{pmatrix} a_{10} & a_{11} & \dots & a_{1,m-1} \\ a_{20} & a_{21} & \dots & a_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n0} & a_{n1} & \dots & a_{n,m-1} \end{pmatrix} \cdot \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{pmatrix} = 0$$

where $n < m$ and the $a_{ij} \in \mathbb{F}_p$ are chosen uniformly at random.

One defines:

- Ring-SIS (assume $m = kn$):
  - Find small $\mathbf{x}_i \in R_p = \mathbb{Z}[x]/(f(x))$ with $\deg f(x) = n$ such that

$$\begin{pmatrix} \mathbf{a}_1 & \dots & \mathbf{a}_k \end{pmatrix} \cdot \begin{pmatrix} \mathbf{x}_1 & \dots & \mathbf{x}_k \end{pmatrix}^T = 0$$

    with $\mathbf{a}_i \in R_p$ random.
- Module-SIS: similar (fill matrix with blocks)

Proven to be at least as hard as —/Ideal/Module-SIVP.

9. Don't push it

Reconsider Module-LWE, where one is to solve a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \cdots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \cdots & \mathbf{a}_{k,\ell-1} \end{pmatrix}}_{\mathbf{A}} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

over R_p^ℓ.

9. Don't push it

Reconsider Module-LWE, where one is to solve a noisy system

$$\begin{pmatrix} \mathbf{b}_0 \\ \mathbf{b}_1 \\ \vdots \\ \mathbf{b}_{k-1} \end{pmatrix} = \underbrace{\begin{pmatrix} \mathbf{a}_{10} & \mathbf{a}_{11} & \cdots & \mathbf{a}_{1,\ell-1} \\ \mathbf{a}_{20} & \mathbf{a}_{21} & \cdots & \mathbf{a}_{2,\ell-1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_{k0} & \mathbf{a}_{k1} & \cdots & \mathbf{a}_{k,\ell-1} \end{pmatrix}}_{\mathbf{A}} \cdot \begin{pmatrix} \mathbf{s}_0 \\ \mathbf{s}_1 \\ \vdots \\ \mathbf{s}_{\ell-1} \end{pmatrix} + \begin{pmatrix} \mathbf{e}_0 \\ \mathbf{e}_1 \\ \vdots \\ \mathbf{e}_{k-1} \end{pmatrix}$$

over R_p^ℓ.

What if we push it, identify key space

$$R_p^\ell \qquad \text{with} \qquad \frac{R[y]}{(p, g(y))}$$

for some monic deg ℓ polynomial $g(y) \in R[y]$, by viewing

$$(\mathbf{s}_0, \mathbf{s}_1, \ldots, \mathbf{s}_{\ell-1}) \quad \text{as} \quad \mathbf{s}_0 + \mathbf{s}_1 y + \mathbf{s}_2 y^2 + \cdots + \mathbf{s}_{\ell-1} y^{\ell-1},$$

and replace $\mathbf{A}$ with $\mathbf{A}_{\mathfrak{a}(y)}$ for random $\mathfrak{a}(y)$?

9. Don't push it

Can be a bad idea:

- [PTP15] suggest to work with

$$f(x) = x^{\ell'} + 1,\ \ell' = 2^{k'}, \qquad g(y) = y^{\ell} + 1,\ \ell = 2^{k},$$

which amounts to working in the ring

$$\frac{\mathbb{Z}[x, y]}{(p, x^{\ell'} + 1, y^{\ell} + 1)}$$

and identifying

$$(s_{00}, s_{01}, \ldots, s_{\ell'-1,\ell-1}) \in \mathbb{F}_p^n \quad \text{with} \quad \sum_{\substack{0 \le i \le \ell' \\ 0 \le j \le \ell}} s_{ij} x^i y^j.$$

9. Don't push it

Can be a bad idea:

- [PTP15] suggest to work with

$$f(x) = x^{\ell'} + 1,\ \ell' = 2^{k'}, \qquad g(y) = y^{\ell} + 1,\ \ell = 2^{k},$$

  which amounts to working in the ring

$$\frac{\mathbb{Z}[x, y]}{(p, x^{\ell'} + 1, y^{\ell} + 1)}$$

  and identifying

$$(s_{00}, s_{01}, \ldots, s_{\ell'-1,\ell-1}) \in \mathbb{F}_p^n \quad \text{with} \quad \sum_{\substack{0 \le i \le \ell' \\ 0 \le j \le \ell}} s_{ij} x^i y^j.$$

- Assume (wlog) that $\ell' \geq \ell$, then $x^{\ell'/\ell}$ is a root of $y^{\ell} + 1$!

9. Don't push it

Can be a bad idea:

- So we have a smallness preserving homomorphism

$$\frac{\mathbb{Z}[x,y]}{(p, x^{\ell'}+1, y^{\ell}+1)} \to \frac{\mathbb{Z}[x]}{(p, x^{\ell'}+1)} : \mathbf{s}(x,y) \mapsto \mathbf{s}(x, x^{\ell'/\ell})$$

  and solving smaller-dim'l Ring-LWE reveals $\mathbf{s}(x, x^{\ell'/\ell})$.
- By varying the roots $x^{\ell'/\ell}, x^{3\ell'/\ell}, x^{5\ell'/\ell}, \ldots$ we retrieve all of $\mathbf{s}(x,y)$ through simple linear algebra.

9. Don't push it

Can be a bad idea:

- So we have a smallness preserving homomorphism

$$\frac{\mathbb{Z}[x,y]}{(p, x^{\ell'}+1, y^{\ell}+1)} \to \frac{\mathbb{Z}[x]}{(p, x^{\ell'}+1)} : \mathbf{s}(x,y) \mapsto \mathbf{s}(x, x^{\ell'/\ell})$$

  and solving smaller-dim'l Ring-LWE reveals $\mathbf{s}(x, x^{\ell'/\ell})$.
- By varying the roots $x^{\ell'/\ell}, x^{3\ell'/\ell}, x^{5\ell'/\ell}, \ldots$ we retrieve all of $\mathbf{s}(x,y)$ through simple linear algebra.

In general:

- if the two ring structures are independent, then essentially reduce to Ring-LWE,
- if the two ring structures are dependent, then suffer from the above reduction.

9. Don't push it

Can be a bad idea:

- So we have a smallness preserving homomorphism

$$\frac{\mathbb{Z}[x,y]}{(p, x^{\ell'}+1, y^{\ell}+1)} \to \frac{\mathbb{Z}[x]}{(p, x^{\ell'}+1)} : \mathbf{s}(x,y) \mapsto \mathbf{s}(x, x^{\ell'/\ell})$$

  and solving smaller-dim'l Ring-LWE reveals $\mathbf{s}(x, x^{\ell'/\ell})$.
- By varying the roots $x^{\ell'/\ell}, x^{3\ell'/\ell}, x^{5\ell'/\ell}, \ldots$ we retrieve all of $\mathbf{s}(x,y)$ through simple linear algebra.

In general:

- if the two ring structures are independent, then essentially reduce to Ring-LWE,
- if the two ring structures are dependent, then suffer from the above reduction.

Does not seem interesting track…

9. Don't push it

Example/remark: consider a Module-NTRU sample

$$H = A_{\mathfrak{a}(y)} A_{\mathfrak{b}(y)}^{-1} \in R_p^{2\times 2}, \qquad R = \frac{\mathbb{Z}[x]}{(x^{n/2}+1)}$$

with matrices of multiplication by

$$\mathfrak{a}(y), \mathfrak{b}(y) \in \frac{R[y]}{(p, y^2 - x)} = \frac{\mathbb{Z}[x,y]}{(p, x^{n/2}+1, y^2 - x)} \cong \frac{\mathbb{Z}[y]}{(p, y^n + 1)}.$$

This becomes a standard Ring-LWE sample.

9. Don't push it

Example/remark: consider a Module-NTRU sample

$$H = A_{\mathfrak{a}(y)} A_{\mathfrak{b}(y)}^{-1} \in R_p^{2\times 2}, \qquad R = \frac{\mathbb{Z}[x]}{(x^{n/2}+1)}$$

with matrices of multiplication by

$$\mathfrak{a}(y), \mathfrak{b}(y) \in \frac{R[y]}{(p, y^2 - x)} = \frac{\mathbb{Z}[x,y]}{(p, x^{n/2}+1, y^2 - x)} \cong \frac{\mathbb{Z}[y]}{(p, y^n + 1)}.$$

This becomes a standard Ring-LWE sample.

Interpretation of the determinant reduction:

$$\det H = \frac{\det A_{\mathfrak{a}(y)}}{\det A_{\mathfrak{b}(y)}} = \frac{N(\mathfrak{a}(y))}{N(\mathfrak{b}(y))}$$

$\rightsquigarrow$ used in [ABD16] to attack overstretched NTRU.

10. Polynomial ciphertext modulus

Return to ring-based LWE and take a step back...

We start with our parent ring $R = \mathbb{Z}[x]/(f(x))$
with $f(x)$ monic of degree n.

10. Polynomial ciphertext modulus

Return to ring-based LWE and take a step back…

We start with our parent ring $R = \mathbb{Z}[x]/(f(x))$
with $f(x)$ monic of degree n.

Note:
- Free $\mathbb{Z}$-module with basis $1, x, \ldots, x^{n-1}$.
- Smallness is defined at this level.
  E.g., coefficients from spherical Gaussian.

10. Polynomial ciphertext modulus

Return to ring-based LWE and take a step back…

We start with our parent ring $R = \mathbb{Z}[x]/(f(x))$
with $f(x)$ monic of degree n.

Note:
- Free $\mathbb{Z}$-module with basis $1, x, \ldots, x^{n-1}$.
- Smallness is defined at this level.
  E.g., coefficients from spherical Gaussian.

Next we quotient out by a ciphertext modulus to end up in

$$R_p = \mathbb{Z}[x]/(p, f(x)) \qquad \mathrm{Rep}(R_p) = \left\{ \sum_{0 \leq i < n} a_i x^i \mid 0 \leq a_i < p \right\}$$

where:
- small elt.'s are reductions mod p of small elt.'s of R
  (easy to recognize when isolated),
- all computations are reduced into $\mathrm{Rep}(R_p)$
  (wrap around $\rightsquigarrow$ hard to recognize in expressions)

10. Polynomial ciphertext modulus

What if we replace p by a polynomial modulus?

- Pick monic polynomial $f(x) \in \mathbb{Z}[x]$ defining the parent ring:

$$R = \mathbb{Z}[x]/(f(x)).$$

10. Polynomial ciphertext modulus

What if we replace p by a polynomial modulus?

- Pick monic polynomial $f(x) \in \mathbb{Z}[x]$ defining the parent ring:

$$R = \mathbb{Z}[x]/(f(x)).$$

- Choose error distribution to define smallness.

10. Polynomial ciphertext modulus

What if we replace p by a polynomial modulus?

- Pick monic polynomial $f(x) \in \mathbb{Z}[x]$ defining the parent ring:
$$R = \mathbb{Z}[x]/(f(x)).$$
- Choose error distribution to define smallness.
- Pick $g(x) \in \mathbb{Z}[x]$ coprime with $f(x)$ and assume that
$$(f(x), g(x)) = (a, r(x)) \qquad \text{for } a \in \mathbb{Z} \text{ and monic } r(x) \in \mathbb{Z}[x]$$
(true for about 60.8% of all polynomial pairs $f(x)$ and $g(x)$).
This gives an easy set of representatives:
$$\mathrm{Rep}(R_{g(x)}) = \left\{ \sum_{0 \le i < \deg r(x)} a_i x^i \mid 0 \le a_i < a \right\}.$$
in which all results are to be reduced.

10. Polynomial ciphertext modulus

What if we replace p by a polynomial modulus?

- Pick monic polynomial $f(x) \in \mathbb{Z}[x]$ defining the parent ring:
$$R = \mathbb{Z}[x]/(f(x)).$$
- Choose error distribution to define smallness.
- Pick $g(x) \in \mathbb{Z}[x]$ coprime with $f(x)$ and assume that
$$(f(x), g(x)) = (a, r(x)) \qquad \text{for } a \in \mathbb{Z} \text{ and monic } r(x) \in \mathbb{Z}[x]$$
(true for about 60.8% of all polynomial pairs $f(x)$ and $g(x)$). This gives an easy set of representatives:
$$\mathrm{Rep}(R_{g(x)}) = \left\{ \sum_{0 \leq i < \deg r(x)} a_i x^i \mid 0 \leq a_i < a \right\}.$$
in which all results are to be reduced.
- Recognizing small elements seems ad hoc exercise.

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.
- Smallness: samples from an extremely narrow Gaussian, so that all coefficients are 0 with just a few ± 1's.

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.
- Smallness: samples from an extremely narrow Gaussian, so that all coefficients are 0 with just a few ± 1's.
- Now quotient out by $x - 2$ to get ciphertext ring

$$R_{x-2} = \frac{\mathbb{Z}[x]}{(x^n - 1, x - 2)} = \frac{\mathbb{Z}[x]}{(2^n - 1, x - 2)} \cong \frac{\mathbb{Z}}{(2^n - 1)}$$

which comes with representatives

$$\mathrm{Rep}(R_{x-2}) = \{\, a \in \mathbb{Z} \mid 0 \leq a < 2^n - 1 \,\}$$

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.
- Smallness: samples from an extremely narrow Gaussian, so that all coefficients are 0 with just a few ± 1's.
- Now quotient out by $x - 2$ to get ciphertext ring

$$R_{x-2} = \frac{\mathbb{Z}[x]}{(x^n - 1, x - 2)} = \frac{\mathbb{Z}[x]}{(2^n - 1, x - 2)} \cong \frac{\mathbb{Z}}{(2^n - 1)}$$

which comes with representatives

$$\mathrm{Rep}(R_{x-2}) = \{\, a \in \mathbb{Z} \,|\, 0 \leq a < 2^n - 1 \,\}$$

- Easy to recognize small elements (Hamming weight)

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.
- Smallness: samples from an extremely narrow Gaussian, so that all coefficients are 0 with just a few ± 1's.
- Now quotient out by $x - 2$ to get ciphertext ring

$$R_{x-2} = \frac{\mathbb{Z}[x]}{(x^n - 1, x - 2)} = \frac{\mathbb{Z}[x]}{(2^n - 1, x - 2)} \cong \frac{\mathbb{Z}}{(2^n - 1)}$$

which comes with representatives

$$\mathrm{Rep}(R_{x-2}) = \{\, a \in \mathbb{Z} \,|\, 0 \leq a < 2^n - 1 \,\}$$

- Easy to recognize small elements (Hamming weight)
- Note that R_{x-2} is totally invulnerable to evaluation-at-1.

10. Polynomial ciphertext modulus

Example:

- Parent ring: $R = \mathbb{Z}[x]/(f(x))$ with $f(x) = x^n - 1$.
- Smallness: samples from an extremely narrow Gaussian, so that all coefficients are 0 with just a few ± 1's.
- Now quotient out by $x - 2$ to get ciphertext ring

$$R_{x-2} = \frac{\mathbb{Z}[x]}{(x^n - 1, x - 2)} = \frac{\mathbb{Z}[x]}{(2^n - 1, x - 2)} \cong \frac{\mathbb{Z}}{(2^n - 1)}$$

  which comes with representatives

$$\mathrm{Rep}(R_{x-2}) = \{\, a \in \mathbb{Z} \mid 0 \leq a < 2^n - 1 \,\}$$

- Easy to recognize small elements (Hamming weight)
- Note that R_{x-2} is totally invulnerable to evaluation-at-1.
- Essentially the Mersenne based system from [AJPS17].

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

2. Select error distribution.

3. Select ciphertext modulus $g(x)$ subject to constraints.

4. Select the rank of the module.

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

   $R = \mathbb{Z} \quad (= \mathbb{Z}[x]/(x))$

2. Select error distribution.

   Gaussian

3. Select ciphertext modulus $g(x)$ subject to constraints.

   $g(x) = p$

4. Select the rank of the module.

   rank n, so work in R_p^n

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

LWE

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

   $R = \mathbb{Z}[x]/(x^n + 1) \quad (n = 2^k)$

2. Select error distribution.

   spherical Gaussian

3. Select ciphertext modulus $g(x)$ subject to constraints.

   $g(x) = p$

4. Select the rank of the module.

   rank 1, so work in R_p

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

Ring-LWE

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

   $$R = \mathbb{Z}[x]/(x^q - x - 1)$$

2. Select error distribution.

   coefficients uniform in $\{0, \pm 1\}$ with fixed weight

3. Select ciphertext modulus $g(x)$ subject to constraints.

   $$g(x) = p$$

4. Select the rank of the module.

   rank 1, so work in R_p

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

NTRU Prime

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

   $R = \mathbb{Z}[x]/(x^D - x^{D/2} - 1)$

2. Select error distribution.

   coefficients sampled from $\{0, \pm 1\}$

3. Select ciphertext modulus $g(x)$ subject to constraints.

   $g(x) = x - 2$

4. Select the rank of the module.

   small rank $n \in \{2, 3, 4\}$, so work in R^n_{x-2}

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

Three Bears (I-MLWE)

11. A recipe for constructing hard problems

1. Select the parent ring $R = \mathbb{Z}[x]/(f(x))$.

   $$R = \mathbb{Z}[x]/(x^n + 1) \quad (n = 2^k)$$

2. Select error distribution.

   spherical binomial

3. Select ciphertext modulus $g(x)$ subject to constraints.

   $$g(x) = p$$

4. Select the rank of the module.

   small rank $n \in \{2, 3, 4\}$, so work in R_p^n

5. Select your hard problem family:

   Module-LWE, Module-NTRU or Module-SIS.

Kyber

Questions?