A Quantum Algorithm for Finding Midly Short Vectors in Cyclotomic Ideal Lattices

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Based on Joint Work with
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Lattice-Based Crypto

Lattice problems provide a strong fundation for Post-Quantum Crypto

Worst-case to average-case reduction [Ajt99, Reg09]

$$\text{Worst-case Approx-SVP} \leq \begin{cases} \text{SIS} & \text{(Short Intreger Solution)} \\ \text{LWE} & \text{(Learning With Errors)} \end{cases}$$

How hard is Approx-SVP ? Depends on the Approximation factor α.

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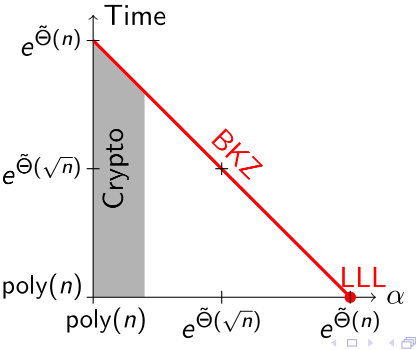

Lattices over Rings (Ideals, Modules)

Generic lattices are cumbersome! Key-size $= \tilde{O}(n^2)$.

NTRU Cryptosystems [HPS98, HHGP+03]

Use the convolution ring $\mathcal{R} = R[X]/(X^p - 1)$, and module-lattices:

$$\mathcal{L}_h = \{(x, y) \in \mathcal{R}^2, \quad hx + y \equiv 0 \bmod q\}.$$

Same lattice dimension, Key-Size $= \tilde{O}(n)$. Later came variants with worst-case fundations:

wc-to-ac reduction [Mic07, LPR13]

$$\text{Worst-case Approx-Ideal-SVP} \leq \begin{cases} \text{Ring-SIS} \\ \text{Ring-LWE} \end{cases}$$

Applicable for cyclotomic rings $\mathcal{R} = \mathbb{Z}[\zeta_m]$ (ζ_m a primitive m-th root of unity).

Denote $n = \deg \mathcal{R}$. In our cyclotomic cases: $n = \phi(m) \sim m$.

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Approx-Ideal-SVP in poly-time for large α

Approx-Ideal-SVP **solvable** in Quantum poly-time, for

$$\mathcal{R} = \mathbb{Z}[\zeta_m], \quad \alpha = \exp(\tilde{O}(\sqrt{n})).$$

Better tradeoffs

Impact and limitations

- No schemes broken
- **Hardness gap** between SVP and Ideal-SVP
- New cryptanalytic tools

$\Rightarrow$ start favoring **weaker assumptions** ? e.g. Module-LWE

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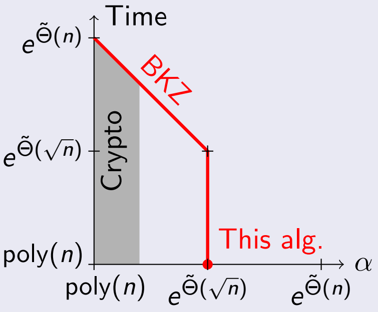

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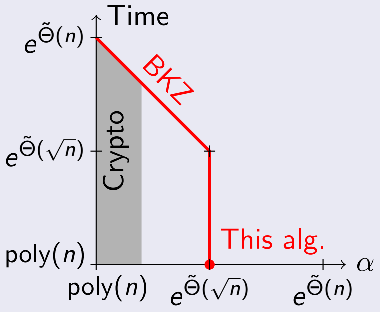

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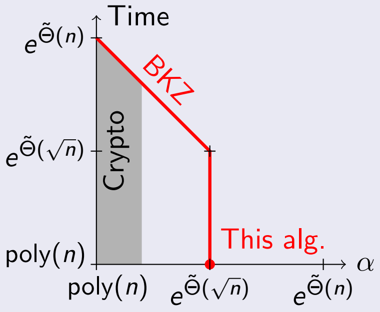

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Algebraic Number Theory

Ideals and Principal Ideals

Cyclotomic number field: $K(= \mathbb{Q}(\zeta_m))$, ring of integer $\mathcal{O}_K(= \mathbb{Z}[\zeta_m])$, where ζ_m is a formal m-th root of unity. The degree of K is $n = \varphi(m)$

Definition (Ideals)

- An **integral ideal** is a subset $\mathfrak{h} \subset \mathcal{O}_K$ closed under addition, and by multiplication by elements of $\mathcal{O}_K$,
- A **(fractional) ideal** is a subset $\mathfrak{f} \subset K$ of the form $\mathfrak{f} = \frac{1}{x}\mathfrak{h}$, where $x \in \mathbb{Z}$,
- A **principal ideal** is an ideal $\mathfrak{f}$ of the form $\mathfrak{f} = g\mathcal{O}_K$ for some $g \in K$.

In particular, ideals are lattices.

We denote $\mathcal{F}_K$ the set of fractional ideals,
and $\mathcal{P}_K$ the set of principal ideals.

Ideals as Lattices

There is a Ring morphism (the empeddings):

$$K \to \mathbb{C}^n$$
$$\zeta \mapsto (\omega^i)_{i \in \mathbb{Z}_m^\times}$$

where $\omega \in \mathbb{C}$ is a complex m-th root of unity. This allows to view ideal as lattices.

- One can therefore view Ideal as lattices
- In the empedding space, multiplication is component-wise $\quad (\simeq \text{FFT})$

Class Group

Ideals can be multiplied, and remain ideals:

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{\text{finite}} a_i b_i, \quad a_i \in \mathfrak{a}, b_i \in \mathfrak{b} \right\}.$$

The product of two principal ideals remains principal:

$$(a\mathcal{O}_K)(b\mathcal{O}_K) = (ab)\mathcal{O}_K.$$

$\mathcal{F}_K$ form an **abelian group**[1], $\mathcal{P}_K$ is a **subgroup** of it.

Definition (Class Group)

Their quotient forms the **class group** $\mathrm{Cl}_K = \mathcal{F}_K/\mathcal{P}_K$.
The class of an ideal $\mathfrak{a} \in \mathcal{F}_K$ is denoted $[\mathfrak{a}] \in \mathrm{Cl}_K$.

An ideal $\mathfrak{a}$ is principal iff $[\mathfrak{a}] = [\mathcal{O}_K]$.

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Lattice of Class Relations

Choose a factor basis: a set $\mathfrak{B} = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ such that $\{[\mathfrak{p}_1], \dots, [\mathfrak{p}_k]\}$ generates Cl_K.
Consider the morphism

$$\phi : \mathbb{Z}^k \to \mathrm{Cl}_K$$
$$(x_1, \dots x_k) \mapsto \left[\prod \mathfrak{p}_i^{x_i}\right]$$

- The kernel $\Lambda = \ker \phi$ is the lattice of class relation over $\mathfrak{B}$.
- Reducing x_i modulo Λ : finding a small representative in the same class

Unit Group and Principal Ideals

- An element $g \in K^\times$ generates an ideal $g\mathcal{O}_K$ $(\approx GL_n(\mathbb{R}))$
- The unit group $\mathcal{O}_K^\times = \{x \in \mathcal{O}_K \text{ s.t. } x^{-1} \in \mathcal{O}_K\}$. $(\approx GL_n(\mathbb{Z}))$
- g and h generates the same ideal iff $g = uh$ for some unit $u \in \mathcal{O}_K$

$$\mathcal{P}_K \simeq K^\times / \mathcal{O}_K^\times$$

$$(\{\text{lattices}\} \simeq GL_n(\mathbb{R})/GL_n(\mathbb{Z}))$$

- Unlike $\{\text{lattices}\} \simeq GL_n(\mathbb{R})/GL_n(\mathbb{Z})$, the groups are *commutative* : reduction should be much easier...
- Spoiler: take the log and $\mathcal{O}_K^\times$ becomes a lattice !

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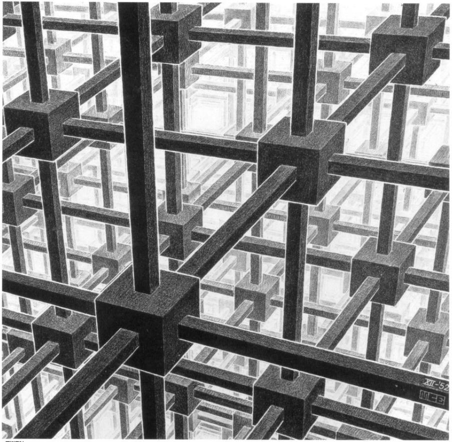
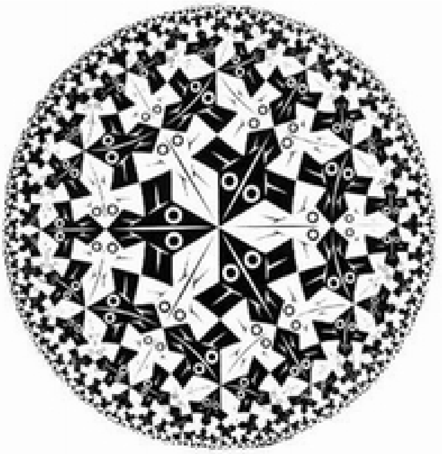
Tesselation: commutative v.s. non-commutative





Overview

4 Steps to Ideal-SVP

The Close Principal Multiple Problem

Given an ideal $\mathfrak{a}$, find $\mathfrak{c} \subset \mathfrak{a}$ that is principal, and not much sparser

1. Find a representative $\prod \mathfrak{p}_i^{x_i}$ of $[\mathfrak{a}^{-1}]$ [EHKS14, BS15]
2. Make the x_i small by reducing modulo class relations [CDW17]

$\Rightarrow$ Output $\mathfrak{c} = \mathfrak{a} \cdot \prod \mathfrak{p}_i^{x_i}$

Short Generator Problem

Given a principal ideal $\mathfrak{c}$, find a short generator g of $\mathfrak{c}$

3. Find any generator g of $\mathfrak{c}$ [EHKS14, BS15]
4. Reduce g modulo the unit group $\mathcal{O}_K^\times$ [CDPR16]

$\Rightarrow$ Output g

This talk

Working Hypothesis

$$|\text{Quantum}\rangle = \mathfrak{Magic}$$

We will focus on the following steps:

2. Make the x_i small by reducing modulo class relations [CDW17]
4. Reduce g modulo the unit group $\mathcal{O}_K^\times$ [CDPR16]

For a survey covering all the steps, refer to [Duc17].

The Close Principal Multiple Problem

From CPM to Ideal-SVP

Definition (The Close Principal Multiple problem)

- Given an ideal $\mathfrak{a}$, and an factor F
- Find a **small integral** ideal $\mathfrak{b}$ such that $[\mathfrak{a}\mathfrak{b}] = [\mathcal{O}_K]$ and $N\mathfrak{b} \leq F$

Smallness is with respect to the Algebraic Norm N of $\mathfrak{b}$,
(essentially the **volume** of $\mathfrak{b}$ as a lattice).

Factor Basis, Class-Group Discrete-Log

Choose a **factor basis** $\mathfrak{B}$ of integral ideals and search $\mathfrak{b}$ of the form:

$$\mathfrak{b} = \prod_{\mathfrak{p} \in \mathfrak{B}} \mathfrak{p}^{e_\mathfrak{p}}.$$

We choose $|\mathfrak{B}|$ small (say n) and $\mathfrak{p} \in \mathfrak{B}$ small as well $N\mathfrak{p} = \mathrm{poly}(n)$.

Corollary (Quantum Cl-Discrete Logarithm,[BS15])

Assume $\mathfrak{B}$ generates the class-group. Given $\mathfrak{a}$ and $\mathfrak{B}$, one can find in quantum polynomial time a vector $\mathbf{e} \in \mathbb{Z}^{\mathfrak{B}}$ such that:

$$\prod_{\mathfrak{p} \in \mathfrak{B}} [\mathfrak{p}^{e_\mathfrak{p}}] = [\mathfrak{a}^{-1}].$$

This finds a $\mathfrak{b}$ such that $[\mathfrak{a}\mathfrak{b}] = [\mathcal{O}_K]$, yet:

- $\mathfrak{b}$ may not be integral (negative exponents, yet easy to solve)
- $N\mathfrak{b} \approx \exp(\|\mathbf{e}\|_1)$ may be huge (unbounded $\mathbf{e}$, want $\|\mathbf{e}\|_1 = \tilde{O}(n^{3/2})$).

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Navigating the Class-Group

Cayley-Graph(G, A):

- A node for any element $g \in G$
- An arrow $g \xrightarrow{a} ga$ for any $g \in G$, $a \in A$

Figure: Cayley-Graph$((\mathbb{Z}/5\mathbb{Z}, +)$,$\{1$,$2\})$

Rephrased Goal for CPM

Find a **short** path from $[\mathfrak{a}]$ to $[\mathcal{O}_K]$ in Cayley-Graph$(\mathrm{Cl}, \mathfrak{B})$.

- Using a few well chosen ideals in $\mathfrak{B}$, Cayley-Graph$(\mathrm{Cl}, \mathfrak{B})$ is an **expander Graph** [JW15]: very short paths exist.
- Finding such short path generically too costly: $|\mathrm{Cl}| > \exp(n)$

A lattice problem

Cl is **abelian** and **finite**, so $\mathrm{Cl} = \mathbb{Z}^{\mathfrak{B}}/\Lambda$ for some lattice Λ:

$$\Lambda = \left\{ \mathbf{e} \in \mathbb{Z}^{\mathfrak{B}}, \quad s.t. \prod [\mathfrak{p}_{\mathfrak{p}}^{e}] = [\mathcal{O}_K] \right\}$$

i.e. the (full-rank) **lattice of class-relations** in base $\mathfrak{B}$.

Figure: $(\mathbb{Z}/5\mathbb{Z}, +) = \mathbb{Z}^{\{1,2\}}/\Lambda$

Rephrased Goal for CPM: CVP in Λ

Find a **short** path from $t \in \mathbb{Z}^{\mathfrak{B}}$ to any lattice point $v \in \Lambda$.

In general: very hard. But for good Λ, with a good basis, can be easy.

Why should we know anything special about Λ ?

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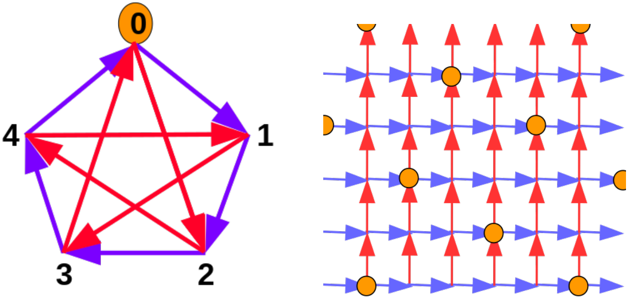
Example

Figure: Cayley-Graph$(\mathbb{Z}/5\mathbb{Z}, \{1,2\}) \quad \simeq \quad \mathbb{Z}^{\{1,2\}}/\Lambda$



More than just a lattice

Let G denote the Galois group, it acts on ideals and therefore on classes:

$$[\mathfrak{a}]^\sigma = [\sigma(\mathfrak{a})].$$

Consider the **group-ring** $\mathbb{Z}[G]$ (formal sums on G), extend the G-action:

$$[\mathfrak{a}]^e = \prod_{\sigma \in G} [\sigma(\mathfrak{a})]^{e_\sigma} \qquad \text{where } e = \sum e_\sigma \sigma.$$

- Assume $\mathfrak{B} = \{\mathfrak{p}^\sigma, \sigma \in G\}$
- G acts on $\mathfrak{B}$, and so it acts on $\mathbb{Z}^{\mathfrak{B}}$ by permuting coordinates
- the lattice $\Lambda \subset \mathbb{Z}^{\mathfrak{B}}$ is **invariant** by the action of G !

i.e. Λ admits G as a group of **symmetries**

Λ is more than just a lattice: it is a $\mathbb{Z}[G]$-**module**

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Stickelberger's Theorem

In fact, we know much more about Λ !

Definition (The Stickelberger ideal)

The **Stickelberger element** $\theta \in \mathbb{Q}[G]$ is defined as

$$\theta = \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^*} \left(\frac{a}{m} \bmod 1 \right) \sigma_a^{-1} \qquad \text{where } G \ni \sigma_a : \omega \mapsto \omega^a.$$

The **Stickelberger ideal** is defined as $S = \mathbb{Z}[G] \cap \theta \mathbb{Z}[G]$.

Theorem (Stickelberger's theorem)

The Stickelberger ideal annihilates the class group: $\forall e \in S, \mathfrak{a} \subset K$

$$[\mathfrak{a}^e] = [\mathcal{O}_K].$$

In particular, if $\mathfrak{B} = \{\mathfrak{p}^\sigma, \sigma \in G\}$, *then* $S \subset \Lambda$.

Geometry of the Stickelberger ideal

Fact

There exists an **explicit** *(efficiently computable)* **short** *basis of S, namely it has ternary coefficients.*

Corollary

Given $t \in \mathbb{Z}[G]$, one can find $x \in S$ suh that $\|x - t\|_1 \leq n^{3/2}$.

Conclusion: back to CPM

Extra technicalities

Convenient simplifications/omissions made so far:

$\mathfrak{B} = \{\mathfrak{p}^\sigma, \sigma \in G\}$ generates the class group.

- can allow a few (say polylog) many different ideals and their conjugates in $\mathfrak{B}$
- Numerical computation says such $\mathfrak{B}$ should exist [Sch98]
- Theorem+Heuristic then say we can find such $\mathfrak{B}$ efficiently

Eliminating minus exponents

- Easy when $h^+ = 1$: $[\mathfrak{a}^{-1}] = [\bar{\mathfrak{a}}]$, doable when $h^+ = \mathrm{poly}(n)$. [a]
- Justified by numerical computations and heuristics

[BPR04, Sch03]

[a] h^+ is the size of the class group of K^+, the max. real subfield of K

The Short Generator Problem

Invocation of $|\text{Quantum}\rangle$ $\mathfrak{Magic}$

Given an ideal $\mathfrak{c}$, one can find a generator h of it using a quantum computer. [EHKS14, Bia14]

The Logarithmic Embedding

The n embeddings $\sigma_i : K \mapsto \mathbb{C}$ for i for $i \in \mathbb{Z}_m^\times$ are given by

$$\sigma_i(\zeta) = \omega^i$$

The logarithmic Embedding is defined as

$$\begin{aligned} \mathrm{Log} : K &\to \mathbb{R}^{n/2} \\ x &\mapsto (\log|\sigma_i(x)|)_{i \in \mathbb{Z}_m^\times/\pm 1} \end{aligned}$$

It induces

- a group morphism from $(K \setminus \{0\}, \cdot)$ to $(\mathbb{R}^{n/2}, +)$
- a monoid morphism from $(R \setminus \{0\}, \cdot)$ to $(\mathbb{R}^n/2, +)$

The Unit Group

By Dirichlet Unit Theorem

- the kernel of Log is the cyclic group T of roots of unity of $\mathcal{O}_K$
- $\operatorname{Log} \mathcal{O}_K^\times \subset \mathbb{R}^n$ is an lattice of rank $r + c - 1$
  (where K has r real embeddings and $2c$ complex embeddings)

Reduction modulo $\mathcal{O}_K$: a Close Vector Problem

Elements $g, h \in K$ generate the same ideal if and only if $h = g \cdot u$ for some unit $u \in \mathcal{O}_K^\times$. In particular

$$\operatorname{Log} g \in \operatorname{Log} h + \operatorname{Log} \mathcal{O}_K^\times.$$

and g is the "smallest" generator iff $\operatorname{Log} u \in \operatorname{Log} \mathcal{O}_K^\times$ is a vector "closest" to $\operatorname{Log} h$.

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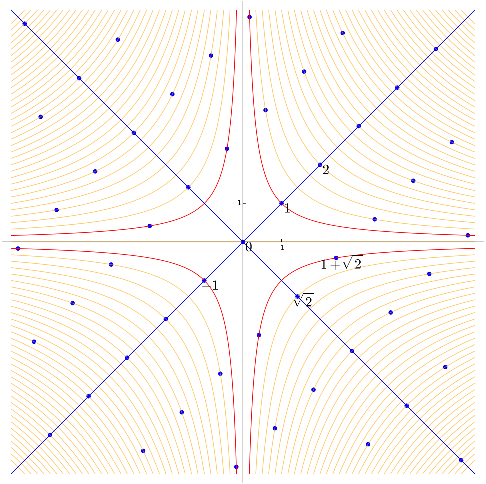
Example: Embedding $\mathbb{Z}[\sqrt{2}] \hookrightarrow \mathbb{R}^2$



- x-axis: $a + b\sqrt{2} \mapsto a + b\sqrt{2}$
- y-axis: $a + b\sqrt{2} \mapsto a - b\sqrt{2}$
- component-wise multiplication
- Symmetries induced by
  - mult. by -1
  - conjugation $\sqrt{2} \mapsto -\sqrt{2}$

"Orthogonal" elements

Units (algebraic norm 1)

"Isonorms" curves

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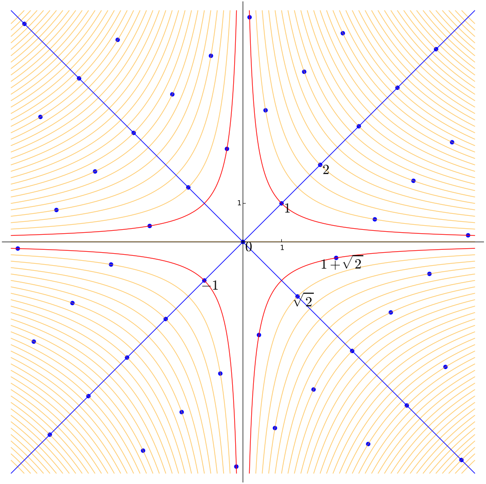

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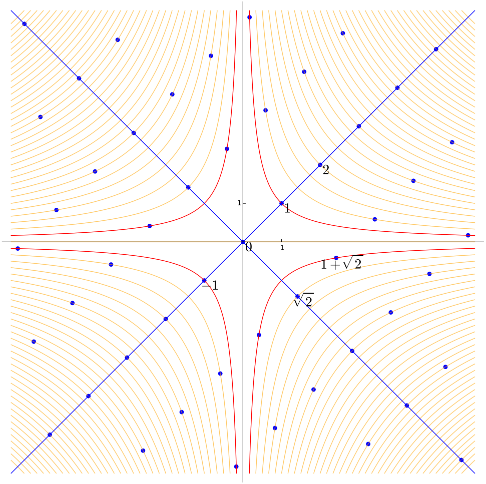

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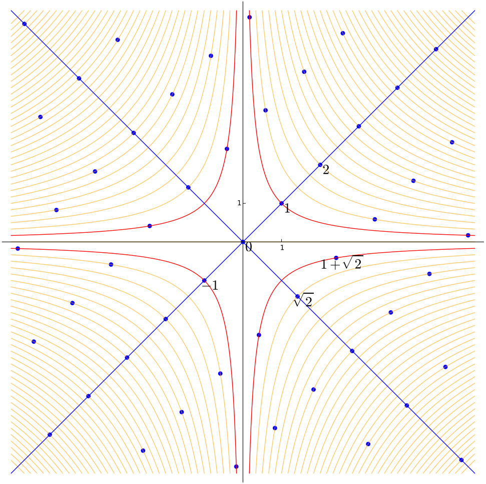

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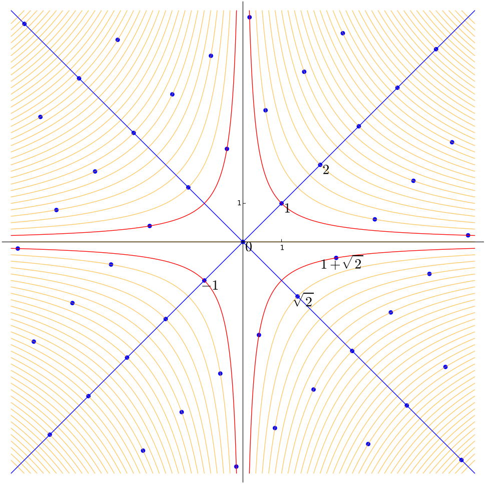

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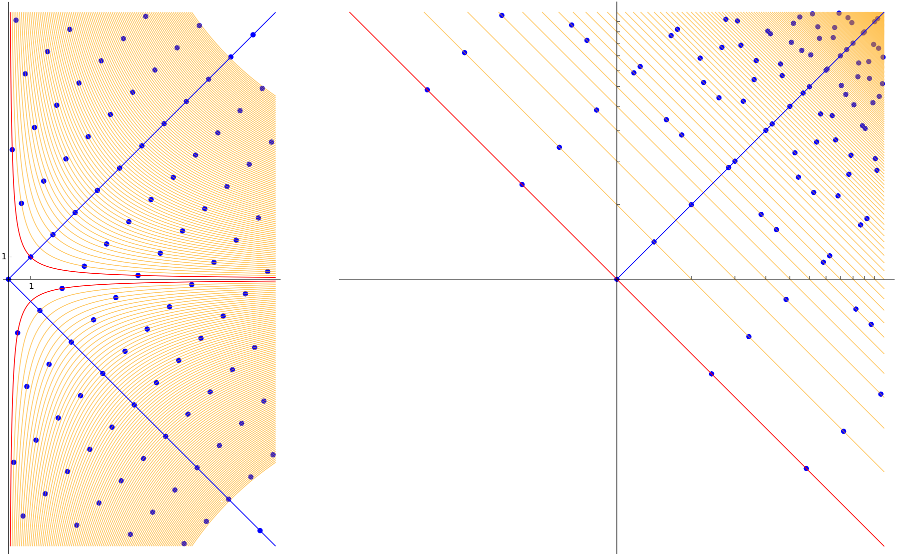
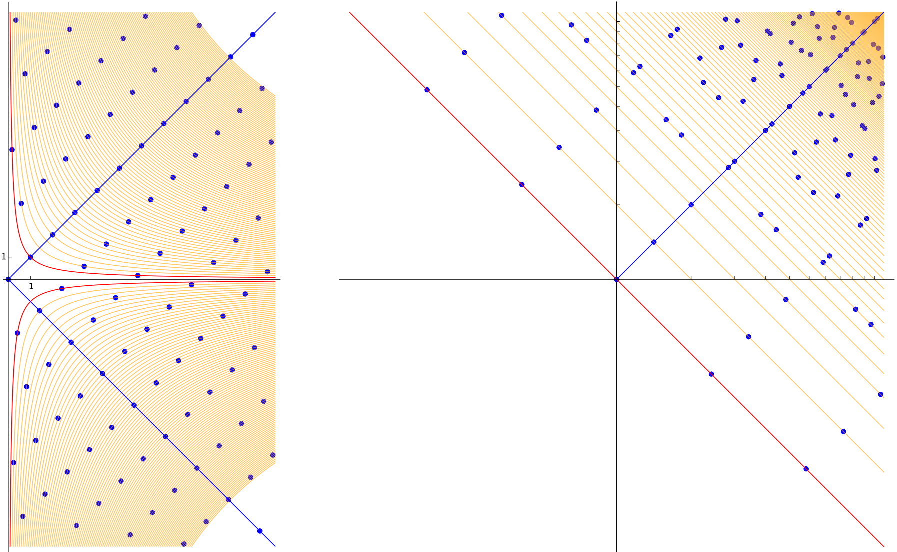
$(\{\bullet\}, +)$ is a sub-monoid of $\mathbb{R}^2$



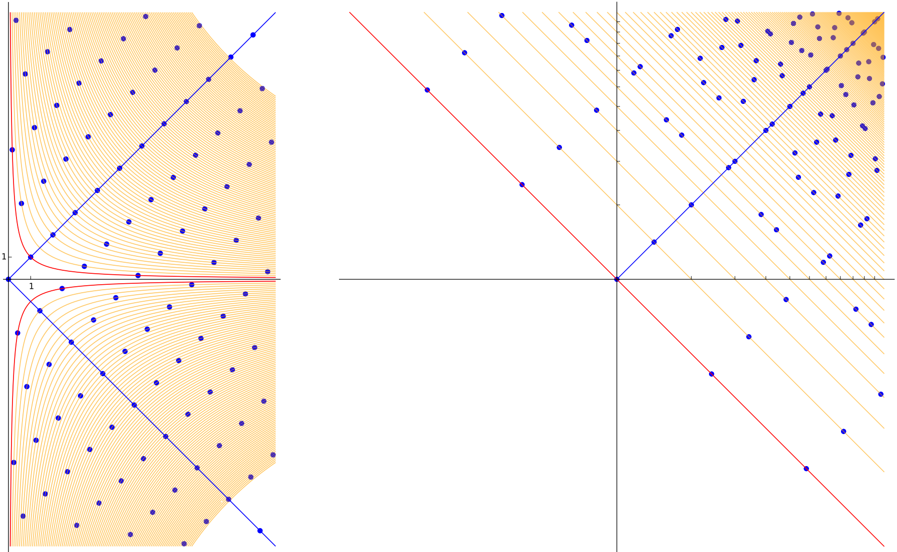
Example: Logarithmic Embedding $\operatorname{Log} \mathbb{Z}[\sqrt{2}]$

$\operatorname{Log} \mathcal{O}_K^\times = (\{\bullet\}, +) \cap \diagdown$ is a lattice of $\mathbb{R}^2$, orthogonal to $(1, 1)$



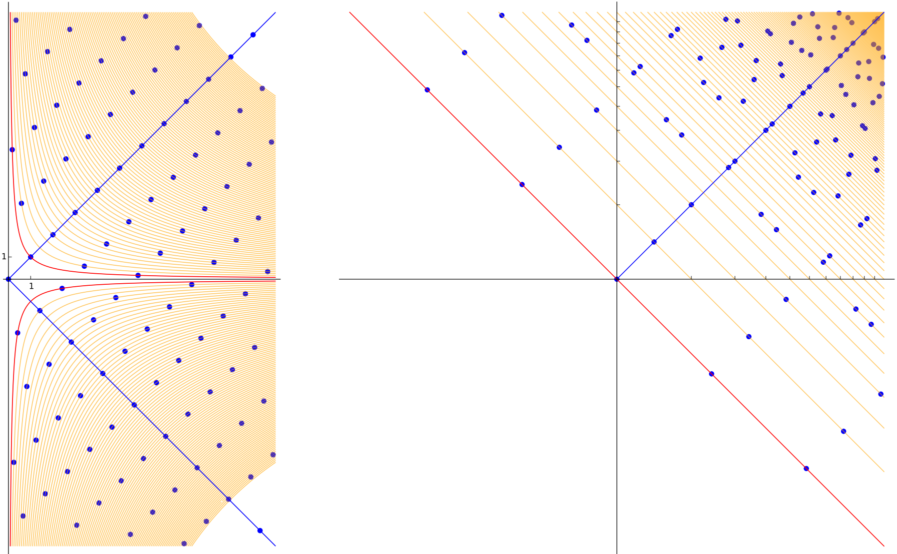
Example: Logarithmic Embedding $\mathrm{Log}\,\mathbb{Z}[\sqrt{2}]$

$\{\bullet\} \cap \diagdown$ are shifted finite copies of $\mathrm{Log}\,\mathcal{O}_K^\times$



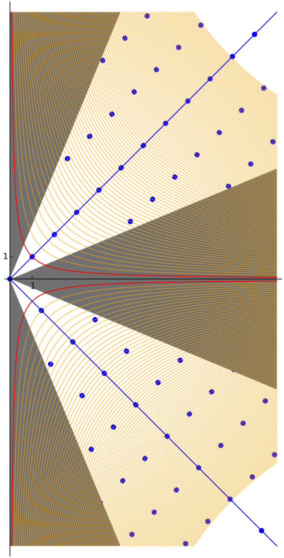
Example: Logarithmic Embedding $\mathrm{Log}\,\mathbb{Z}[\sqrt{2}]$

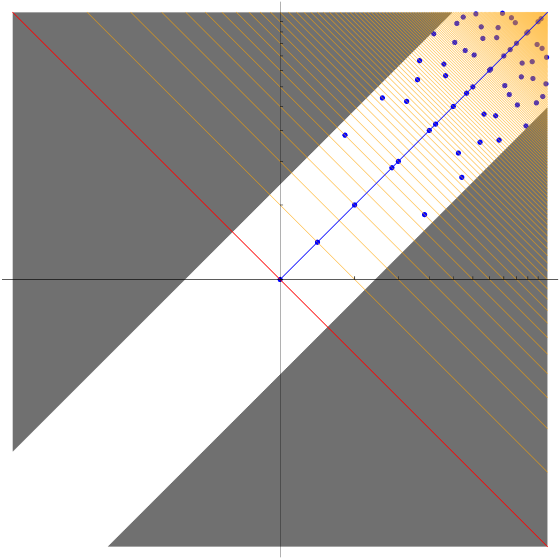
Some $\{\bullet\} \cap$ ╲ may be empty (e.g. no elements of Norm 3 in $\mathbb{Z}[\sqrt{2}]$)



Reduction modulo Log $\mathbb{Z}[\sqrt{2}]^\times$

The reduction modulo $\mathbb{Z}[\sqrt{2}]^\times$.





Cyclotomic units

Let's assume $m = p^k$ for some prime p.

$$z_j = 1 - \zeta^j \quad \text{and } b_j = z_j / z_1 \text{ for all } j \text{ coprimes with } m.$$

The b_j are units, and the group C generated by

$$\zeta, \quad b_j \quad \text{for } j = 2, \ldots m/2, j \text{ coprime with } m$$

is known as the group of *cyclotomic units*.

Simplification 1 (Weber's Class Number Problem)

We assume[2] that $\mathcal{O}_K^\times = C$. It is conjectured to be true for $m = 2^k$.

Simplification 2 (for this talk)

We study the dual matrix $\mathbf{Z}^\vee$, where $\mathbf{z}_j = \operatorname{Log} z_j$.
It can be proved to close to $\mathbf{B}^\vee$ where $\mathbf{b}_j = \mathbf{z}_j - \mathbf{z}_1$.

[2] One just need the index $[\mathcal{O}_K^\times : C] = h^+(m)$ to be small.

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How good is this basis

- It is quite easy to prove that $\|\mathbf{z}_i\| \leq O(\sqrt{m})$.
- $\Rightarrow$ One can solve CVP with ℓ_∞ distance $\leq O(\sqrt{n} \log n)$.
- $\Rightarrow$ we can find a generator of length $\|g\| \leq \exp(O(\sqrt{n} \log n)) \cdot (N\mathfrak{c})^{1/n}$.

QED

Recall that the principal ideal $\mathfrak{c} \subset \mathfrak{a}$ verified $N\mathfrak{c} \leq \exp(n^{3/2}) N\mathfrak{a}$. That gives $g \in \mathfrak{a}$:

$$\|g\| \leq \exp(\sqrt{n} \log n) \cdot \mathrm{vol}(\mathfrak{a})^{1/n}.$$

We have solved Ideal-SVP with approximation fact $\exp(O(\sqrt{n} \log n))$

Don't leave !

More fun with $\operatorname{Log} \mathcal{O}_K^\times$...

How well can we solve BDD in this lattice ?

This actually has devastating consequence for 'atypical' crypto schemes (Soliloquy and the first generation of Fully Homomorphic Encryption Scheme)

Round-Off Decoding

We also need the fundamental domain to have an efficient reduction algorithm. The simplest one follows:

$\textsc{RoundOff}(\mathbf{B}, \mathbf{t})$ for $\mathbf{B}$ a basis of Λ

- Return $\mathbf{B} \cdot \lfloor \mathbf{B}^{-1} \cdot \mathbf{t} \rceil$.

Used as a decoding algorithm, its correctness is characterized by the error $\mathbf{e}$ and the *dual basis* $\mathbf{B}^\vee = \mathbf{B}^{-T}$.

Fact

Suppose $\mathbf{t} = \mathbf{v} + \mathbf{e}$ for some $\mathbf{v} \in \Lambda$. If $\langle \mathbf{b}_j^\vee, \mathbf{e} \rangle \in [-\frac{1}{2}, \frac{1}{2})$ for all j, then

$$\textsc{Round}(\mathbf{B}, \mathbf{t}) = \mathbf{v}.$$

Dual of a Circulant Basis

Notice that $\mathbf{Z}_{ij} = \log|\sigma_j(1-\zeta^i)| = \log|1-\omega^{ij}|$:
the matrix $\mathbf{Z}$ is G-circulant for the cyclic group $G = \mathbb{Z}_m^\times / \pm 1$.

Fact

If $\mathbf{M}$ is a non-singular, G-circulant matrix, then

- *its eigenvalues are given by $\lambda_\chi = \sum_{g\in G} \overline{\chi(g)} \cdot \mathbf{M}_{1,g}$ where $\chi \in \widehat{G}$ is a character $G \to \mathbb{C}$*
- *All the vectors of $\mathbf{M}^\vee$ have the same norm $\|\mathbf{m}_i^\vee\|^2 = \sum_{\chi\in\widehat{G}} |\lambda_\chi|^{-2}$*

Note: The characters of G can be extended to even Dirichlet characters mod m: $\chi : \mathbb{Z} \to \mathbb{C}$, by setting $\chi(a) = 0$ if $\gcd(a, m) > 1$.

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Computing the Eigenvalues

We wish to give a lower bound on $|\lambda_\chi|$ where

$$\lambda_\chi = \sum_{a \in G} \overline{\chi(a)} \cdot \log |1 - \omega^a|.$$

We develop using the Taylor series

$$\log |1 - x| = - \sum_{k \geq 1} x^k / k$$

and obtain

$$-\lambda_\chi = \sum_{a \in G} \sum_{k \geq 1} \overline{\chi(a)} \cdot \frac{\omega^{ka}}{k}.$$

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Computing the Eigenvalues (continued)

We were trying to lower bound $|\lambda_\chi|$ where

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Fact (Separability of Gauss Sums)

If χ is a primitive Dirichlet character $\bmod m$ *then*

$$\sum_{a \in \mathbb{Z}_m^\times} \overline{\chi(a)} \cdot \omega^{ka} = \chi(k) \cdot G(\chi) \quad \text{where } |G(\chi)| = \sqrt{m}.$$

For this talk, let's ignore non-primitive characters. We rewrite

$$|\lambda_\chi| = \sqrt{\frac{m}{2}} \cdot \left| \sum_{k \geq 1} \frac{\chi(k)}{k} \right|.$$

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The Analytic Hammer

We were trying to lower bound $|\lambda_\chi| = \sqrt{\frac{m}{2}} \cdot \left|\sum_{k\geq 1} \frac{\chi(k)}{k}\right|$.
One recognizes a Dirichlet L-series

$$L(s,\chi) = \sum \frac{\chi(k)}{k^s}.$$

Theorem ([Lit24, LLS15])

For any primitive non-quadratic Dirichlet character $\chi \bmod m$ it holds that

$$1/\ell(m) \leq |L(1,\chi)| \leq \ell(m) \qquad \textit{where } \ell(m) = C \ln m$$

for some universal constant $C > 0$.

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One recognizes a Dirichlet L-series

$$L(s,\chi) = \sum \frac{\chi(k)}{k^s}.$$

Theorem ([Lit24, LLS15])

For any primitive non-quadratic Dirichlet character χ mod m *it holds that*

$$1/\ell(m) \leq |L(1,\chi)| \leq \ell(m) \qquad \textit{where } \ell(m) = C \ln m$$

for some universal constant $C > 0$.

Conclusion

Theorem

Then, all the vectors of $\mathbf{B}^\vee$ *have the same norm and, this norm is upper bounded as follows*

$$\left\|\mathbf{b}_j^\vee\right\|^2 \leq O\left(m^{-1} \cdot \log^3 m\right).$$

Further work

D., Plancon, Wesolowski 2019

- Analyse the hidden factors behind $\tilde{O}$'s.
- Predict when this algorithm outperform LLL and BKZ

Hanrot, Stehle and Pellet–Mary 2019

- Using some precomputation depending only on the number field K
- Generalize this results to any number field K
- Generalize to a time/approx-factor trade-off

$$T = \exp(\tilde{O}(n^c)), \alpha = \exp(\tilde{O}(n^{(1-c)/2}))$$

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