Isogeny-based cryptography: cryptanalysis results

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Isogeny-based cryptography

- Recently proposed for post-quantum cryptography
- Natural problems from a number theory point of view
- Some history, e.g. David Kohel's PhD thesis in 1996
- Some still exponential time, even for quantum computers

Hard problems ?

- Isogeny computation problem (CGL hash, CSIDH) :
  Given two randomly chosen isogenous elliptic curves, compute an isogeny between them.
- Supersingular Isogeny Diffie-Hellman protocol :
  Let p a prime such that $2^e 3^f | (p-1)$ and $2^e \approx 3^f \approx \sqrt{p}$. Given two supersingular elliptic curves E_0, E_1 over $\mathbb{F}_{p^2}$ connected by an isogeny $\varphi : E_0 \to E_1$ of degree 2^e, and given the action of φ on the 3^f-torsion, compute φ.

Outline

Computing isogenies (generic supersingular case)

Supersingular isogeny key exchange protocol

Computing isogenies (ordinary curves and CSIDH)

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Charles-Goren-Lauter hash function [CGL08]

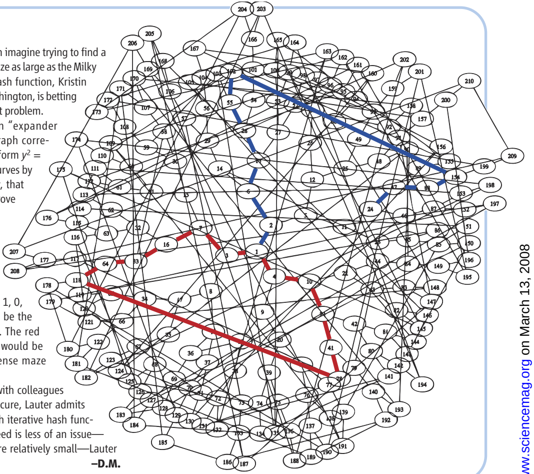
Hash of the Future?

Have you ever struggled to solve a maze? Then imagine trying to find a path through a tangled, three-dimensional maze as large as the Milky Way. By incorporating such a maze into a hash function, Kristin Lauter of Microsoft Research in Redmond, Washington, is betting that neither you nor anyone else will solve that problem.

Technically, Lauter's maze is called an "expander graph" (see figure, right). Nodes in the graph correspond to elliptic curves, or equations of the form $y^2 = x^3 + ax + b$. Each curve leads to three other curves by a mathematical relation, now called isogeny, that Pierre de Fermat discovered while trying to prove his famous Last Theorem.

To hash a digital file using an expander graph, you would convert the bits of data into directions: 0 would mean "turn right," 1 would mean "turn left." In the maze illustrated here, after the initial step 1-2, the blue path encodes the directions 1, 0, 1, 1, 0, 0, 0, 0, 1, ending at point 24, which would be the digital signature of the string 101100001. The red loop shows a collision of two paths, which would be practically impossible to find in the immense maze envisioned by Lauter.

Although her hash function (developed with colleagues Denis Charles and Eyal Goren) is provably secure, Lauter admits that it is not yet fast enough to compete with iterative hash functions. However, for applications in which speed is less of an issue—for example, where the files to be hashed are relatively small—Lauter believes it might be a winner. **–D.M.**



Charles-Goren-Lauter hash function (2)

- Suggested parameters :
  - Supersingular curves (for optimal expansion properties)
  - ℓ isogeny-graph with $\ell = 2$ (for efficiency)
  - "Special" starting curve E_0 (typically $j = 1728$) with known endomorphism ring (no convenient way to select a "random" starting curve)
- Collision, preimage, second preimage resistance naturally translate into isogeny problems, where isogeny degrees are required to be ℓ^e for some e
  - Preimage $\approx$ isogeny $\approx$ path between two vertices
  - Collision $\approx$ endomorphism $\approx$ cycle in the graph

The endomorphism ring of a supersingular curve

- The endomorphism ring of a supersingular curve is a maximal order in the quaternion algebra $B_{p,\infty}$
- In fact, **Deuring correspondence** [D31] : bijection from supersingular curves over $\mathbb{F}_{p^2}$ (up to Galois conjugacy) to maximal orders in $B_{p,\infty}$ (up to conjugation)

$$E \to O \approx \mathrm{End}(E)$$

- Under this correspondence, an isogeny $\phi : E_0 \to E_1$ corresponds to a left ideal of $O_0 \approx \mathrm{End}(E_0)$ which is also a right ideal of $O_1 \approx \mathrm{End}(E_1)$

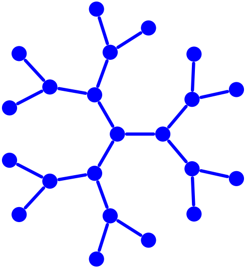
Strategy to break CGL hash function [PL17]

- Idea : given two curves E_0 and E_1
  1. Compute $\mathrm{End}(E_0)$ and $\mathrm{End}(E_1)$
  2. Translate collision and preimage resistance properties from the elliptic curve setting to the quaternion setting
  3. Break collision and preimage resistance for quaternions
  4. Translate the attacks back to elliptic curve setting

- Results so far (1)
  - Breaking CGL hash function (for randomly chosen E_0) is equivalent to computing endomorphism rings

Core problem : computing endomorphisms

- Kohel's algorithm [K96] : fix a small ℓ. Given a curve E, compute all its neighbors in isogeny graph. Compute all neighbors of neighbors, etc, until a loop is found, corresponding to an endomorphism



- Complexity $\tilde{O}(\sqrt{p})$

Some variants

- To compute an isogeny between two curves, grow two trees until a collision is found
- Delfs-Galbraith [DG16] : first compute isogenies to two $\mathbb{F}_p$ curves, then connect those two curves
- Time-memory trade-offs (van Oorschot-Wiener) [vOW94]
- Quantum speedups (?) : cube root quantum claw finding, but may not be practical [JS19]

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- Results so far (2)
  - 2-4 can be solved in polynomial time (modulo heuristics)
  - When a "special" E_0 is chosen in CGL hash function, we can compute collisions in polynomial time
  - Explicit Deuring correspondence easy in one direction (given O compute corresponding j)

Key tools

- Algorithms to solve quaternion norm equations [KLPT14]
- Translation between quaternion ideals and isogenies [W69]
  - Let E_0 with known $\mathrm{End}(E_0) \approx O_0 \subset B_{p,\infty}$
  - Isogenies from E_0 correspond to left ideals of O_0
  - Correspondence computed by identifying kernels
  - Efficient for *powersmooth* norms/degrees
- "Quaternion ℓ-isogeny algorithm" [KLPT14,GPS17]
  - Replace ideal by equivalent one with powersmooth norm

Partial attack on CGL hash function [PL17]

- Suppose CGL hash function uses a **special curve** E_0
- Goal : compute an endomorphism of E_0 of degree ℓ^e (this gives a collision with the void message)
- Compute $\alpha \in O_0 \approx \mathrm{End}(E_0)$ of norm ℓ^e (as in [KLPT14])
- Deduce a collision path in the quaternion setting $I_i = O_0\ell^i + O_0\alpha,\ i = 1, \ldots, e,$ where $n(I_i) = \ell^i$
- For each i
  - Compute $J_i \approx I_i$ with powersmooth norm
  - Compute corresponding isogeny $\varphi_i : E_0 \to E_i$
- Deduce a collision path $(E_0, E_1, \ldots, E_e = E_0)$

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- Results so far
  - Breaking CGL hash function for randomly chosen E_0 is equivalent to computing endomorphism rings
  - When a “special” E_0 is chosen in CGL hash function, we can compute collisions in polynomial time

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Diffie-Hellman key agreement

- Choose g generating a cyclic group
- Alice picks a random a and sends g^a
- Bob picks a random b and sends g^b
- Alice computes $(g^b)^a = g^{ab}$
- Bob computes $(g^a)^b = g^{ab}$
- Eve cannot compute a, b or g^{ab} from g^a and g^b (discrete logarithm, Diffie-Hellman problems)

Isogeny-based Diffie-Hellman [JdF11]

- Choose a prime p, and $N_A, N_B \in \mathbb{N}$ with $\gcd(N_A, N_B) = 1$
  Choose E_0 a supersingular curve over $\mathbb{F}_{p^2}$
- Alice picks a cyclic subgroup $G_A \subset E_0[N_A]$ defining an isogeny $\phi_A : E_0 \to E_A = E_0/G_A$ and she sends E_A to Bob
- Bob picks a cyclic subgroup $G_B \subset E_0[N_B]$ defining an isogeny $\phi_B : E_0 \to E_B = E_0/G_B$ and he sends E_B to Alice

$$\begin{tikzcd}
 & E_A = E_0/G_A \arrow[dr, "\phi'_B"] & \\
E_0 \arrow[ur, "\phi_A"] \arrow[dr, "\phi_B"'] & & E_0/\langle G_A, G_B \rangle \\
 & E_B = E_0/G_B \arrow[ur, "\phi'_A"] &
\end{tikzcd}$$

- Shared key is $E_0/\langle G_A, G_B \rangle = E_B/\phi_B(G_A) = E_A/\phi_A(G_B)$

Isogeny-based Diffie-Hellman (2)

- To compute the shared key Alice will need $\phi_B(G_A)$. This is achieved as follows :
  - Let $G_A = \langle \alpha_A P_A + \beta_A Q_A \rangle$ where $\langle P_A, Q_A \rangle = E_0[N_A]$ and at least one of α_A, β_A coprime to N_A
  - Bob reveals $\phi_B(P_A)$ and $\phi_B(Q_A)$ in addition to E_B
  - Alice computes $\phi_B(G_A) = \langle \alpha_A \phi_B(P_A) + \beta_A \phi_B(Q_A) \rangle$
- Can represent ϕ_A efficiently if N_A smooth
- Can represent torsion points efficiently if either
  - $N_A \mid p - 1$
  - $N_A = \prod \ell_i^{e_i}$ with $\ell_i^{e_i}$ small

Supersingular key agreement protocol [JdF11]

$$P_A, Q_A, R_A$$
$$P_B, Q_B, R_B$$

$$\begin{tikzcd}
 & E_0/\langle R_A\rangle \arrow[dr, "\phi_B'"] & \\
E_0 \arrow[ur, "\phi_A"] \arrow[dr, "\phi_B"'] & & E_0/\langle R_A, R_B\rangle \\
 & E_0/\langle R_B\rangle \arrow[ur, "\phi_A'"'] &
\end{tikzcd}$$

$$\phi_A(P_B), \phi_A(Q_B)$$
$$\phi_A(R_B)$$

$$\phi_B(P_A), \phi_B(Q_A)$$
$$\phi_B(R_A)$$

- Jao-De Feo chose $N_i = \ell_i^{e_i}$ and $p = N_A N_B f + 1$
- A priori safer to use arbitrary primes and $N_i \approx p^2$

Special isogeny problems

- In Jao-De Feo-Plût protocols special problems are used
  1. A special prime p is chosen so that $p = N_1 N_2 \pm 1$ with $N_1 \approx N_2 \approx \sqrt{p}$
  2. There are $\approx p/12$ supersingular invariants but only $N_1 \approx \sqrt{p}$ possible choices for E_1
  3. **Extra information provided** : compute $\phi : E_0 \to E_1$ of degree N_1 **knowing** $\phi(P)$ **for all** $P \in E_0[N_2]$
- Point 2 improves tree-based attacks to $O(p^{1/4})$ (and similar improvements using van Oorschot-Wiener)
- We now focus on Point 3

Impact of torsion points

- Attack on Jao-De Feo-Plût protocol : compute an isogeny $\phi_1 : E_0 \to E_1$ of degree N_1 **given action of ϕ_1 on $E_0[N_2]$**
- How useful is this additional information ?
  - If $d = \gcd(N_1, N_2) \neq 1$ we can recover (part of) ϕ_1
    - Write $\phi_1 = \phi_1' \circ \phi_d$ with $\deg \phi_d = d$
    - Solve DLP modulo d to recover $\ker \phi_d$ hence ϕ_d
    - Find ϕ_1' with a meet-in-the-middle approach
  - In SIDH we have $\gcd(N_1, N_2) = 1$ by design
- Useless ?

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- Using torsion points : active attacks
- Using torsion points : passive attacks

Computing isogenies (ordinary curves and CSIDH)

Active attacks on static keys (1)

- Attack idea : trick Alice into computing $\phi_A(P_B), \phi_A(Q_B)$ for P_B, Q_B of order N_A instead of N_B
  - P_B, Q_B part of a (maliciously generated) public key
  - Fault attack during Alice's computation [T17,GW17]
- More generally, P_B, Q_B of order not coprime with N_B
- Countermeasure : check that P_B, Q_B have order N_B

Active attacks on static keys (2) [GPST16]

- Attack model
  - Alice is using a *static* key α defining a cyclic subgroup $G_A = \langle P_A + \alpha Q_A \rangle \subset E_0[N_A]$
  - Instead of sending $\phi_B(P_A), \phi_B(Q_A)$ as expected, Bob adaptively chooses and sends P_i, Q_i
  - Bob learns whether this modifies the shared key $j(E_B/\langle P_i + \alpha Q_i \rangle) \stackrel{?}{=} j(E_B/\langle \phi_B(P_A) + \alpha \phi_B(Q_A) \rangle)$
  - Bob progressively recovers α with several P_i, Q_i
- Additional constraint : make sure P_i, Q_i look as expected
  - $N_B P_i = N_B Q_i = O$
  - $e_{N_B}(P_i, Q_i) = e_{N_B}(\phi_A(P_B), \phi_A(Q_B)) = e_{N_B}(P_B, Q_B)^{N_A}$

Solution and countermeasure

- Solution for $N_A = 2^e$:
  - Let $\alpha = A_i + 2^i \alpha'$
  - Replace $\phi_A(P_B), \phi_A(Q_B)$ by P_i, Q_i such that

$$\begin{pmatrix} P_i \\ Q_i \end{pmatrix} = \frac{1}{\lambda_i} \begin{pmatrix} 1 & -2^{n-i-1}A_i \\ 0 & 1+2^{n-i-1} \end{pmatrix} \begin{pmatrix} \phi_A(P_B) \\ \phi_A(Q_B) \end{pmatrix}$$

    where $\lambda_i^2 = 1 + 2^{n-i-1} \bmod 2^n$
  - We then have $\langle P_i + \alpha Q_i \rangle = \langle \phi_A(P_B) + \alpha \phi_A(Q_B) \rangle$ iff $2^{n-i-1}(-A_i + \alpha) \bmod 2^n$

- Countermeasure : Fujisaki-Okamoto transform (factor 2 slowdown)

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Impact of torsion points

- Attack on Jao-De Feo-Plût protocol : compute an isogeny $\phi_1 : E_0 \to E_1$ of degree N_1 **given action of** ϕ_1 **on** $E_0[N_2]$
- How useful is this additional information ?
  - If $d = \gcd(N_1, N_2) \neq 1$ we can recover (part of) ϕ_1, but in SIDH we have $\gcd(N_1, N_2) \neq 1$ by design
  - Some active attacks can exploit torsion points
- What about passive attacks ? (honest users)

Warm-up : computing endomorphisms with auxilliary information

- Let p be a prime and let E be a supersingular elliptic curve defined over $\mathbb{F}_{p^2}$. Let ϕ be a non scalar endomorphism of E with smooth order N_1. Let N_2 be a smooth integer with $\gcd(N_1, N_2) = 1$, and let P, Q be a basis of $E[N_2]$.
- Let R be a subring of $\mathrm{End}(E)$ that is either easy to compute, or given (for example, scalar multiplications).
- Given E, P, Q, $\phi(P)$, $\phi(Q)$, $\deg \phi$, R, compute ϕ.

- Best previous algorithm : meet-in-the-middle in $\tilde{O}(\sqrt{N_1})$

Algorithm sketch (with $R = \mathbb{Z}$)

- We know ϕ on the N_2 torsion.
  Deduce $\hat{\phi}$ on the N_2 torsion and $\mathrm{Tr}(\phi)$ if $N_2 > 2\sqrt{N_1}$.
- Consider $\psi := a\phi + b$ for $a, b \in \mathbb{Z}$.
  Can evaluate ψ on the N_2 torsion.
- Find $a, b \in \mathbb{Z}$ such that

$$\deg \psi = a^2 \deg \phi + b^2 + ab\mathrm{Tr}\phi = N_2 N_1'$$

  with N_1' small and smooth. Write $\psi = \psi_{N_1'} \psi_{N_2}$.
- Identify $\ker \psi_{N_2}$ from $\psi(E[N_2])$ and deduce ψ_{N_2}.
- Find $\psi_{N_1'}$ with a meet-in-the-middle strategy.
- Find $\ker \phi$ by evaluating $(\psi - b)/a$ on the N_1 torsion, and deduce ϕ.

Finding (a, b) and Complexity

- We have $$\begin{aligned}\deg\psi &= a^2\deg\phi + b^2 + ab\mathrm{Tr}\phi\\ &= \left(b + a\tfrac{\mathrm{Tr}\phi}{2}\right)^2 + a^2\left(\deg\phi - \left(\tfrac{\mathrm{Tr}\phi}{2}\right)^2\right)\end{aligned}$$
- We want $\deg\psi = N_2 N_1'$ and N_1' small and smooth
- Solutions to $\deg\psi = 0 \bmod N_2$ form a dimension 2 lattice
- We compute a reduced basis, then search for a small linear combination of short vectors until N_1' smooth
- Heuristic analysis shows we can expect $N_1' \approx \sqrt{N_1}$. Revealing $\phi(E[N_2])$ leads to a near square root speedup. (Some parameter restrictions apply.)

Computing isogenies with auxilliary information

- Let p be a prime. Let $N_1, N_2 \in \mathbb{Z}$ coprime. Let E_0 be a supersingular elliptic curve over $\mathbb{F}_{p^2}$. Let $\phi_1 : E_0 \to E_1$ be an isogeny of degree N_1.
- Let R_0, R_1 be subrings of $\mathrm{End}(E_0)$, $\mathrm{End}(E_1)$ respectively. Assume R_0 contains more than scalar multiplications.
- Given N_1, E_1, R_0, R_1 and the image of ϕ_1 on the whole N_2 torsion, compute ϕ_1.

- Best previous algorithm : meet-in-the-middle in $\tilde{O}(\sqrt{N_1})$

General idea

- For $\theta \in \mathrm{End}(E_0)$ consider $\phi = \phi_1 \theta \hat{\phi}_1 \in \mathrm{End}(E_1)$
- Evaluate ϕ on the N_2 torsion
- Apply techniques from above on ϕ
- Compute $\ker \phi \cap E_1[N_1]$
- Deduce $\ker \hat{\phi}_1$, then $\hat{\phi}_1$ and ϕ_1

Remarks

- Several authors have suggested to use $j(E_0) = 1728$ for efficiency reasons. In this case $\mathrm{End}(E_0)$ is entirely known and moreover it contains a degree 1 non scalar element θ. Both aspects are useful in attacks.
- The paper develops two attacks but we expect variants and improvements to come.

Impact on Key Agreement Protocol

- For $j(E_0) = 1728$ and when $N_1 \approx p$ and $N_2 \approx N_1^4$ this approach leads to polynomial time key recovery (heuristic analysis)
- Assuming only that $\mathrm{End}(E_0)$ has a small element, then if $\log N_2 \approx (\log^2 N_1)$, a variant of the above strategy also leads to polynomial time key recovery (heuristic analysis)
- Parameters suggested by De Feo-Jao-Plût $N_1 \approx N_2 \approx \sqrt{p}$ are not affected so far

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Endomorphism ring computation

- Ordinary case : subexponential time (Bisson-Sutherland)
- CSIDH [CLMPR18] : we have

$$\mathbb{Z}[\pi] \subseteq \mathrm{End}(E) \subseteq \mathbb{Z}\left[\frac{\pi+1}{2}\right]$$

where $\pi : (x, y) \to (x^p, y^p)$

and we can easily verify whether $\frac{\pi+1}{2} \in \mathrm{End}(E)$

Computing isogenies : classical algorithms

- We expect $O(p^{1/2})$ supersingular curves over $\mathbb{F}_p$
- Meet-in-the-middle approach restricted to these curves has cost $O(p^{1/4})$

Computing isogenies : quantum algorithms

- Reduction to hidden shift problem : let E_0, E_1 two isogenous curves. For $s \in \mathcal{C}\ell(\mathrm{End}(E_0))$ such that $E_1 = s * E_0$ we have

  $$f(x) = g(xs)$$

  where $f(x) = x * E_1$ and $g(x) = x * E_0$

- Kuperberg's quantum algorithm (or variants) solves this in subexponential time

SIDH vs CSIDH ?

- CSIDH
  - No torsion points revealed
  - Subexponential quantum attacks
  - Still exponential classical attacks
- SIDH
  - Torsion points revealed
    (leading to attacks on overstreched parameters)
  - Still exponential classical and quantum attacks

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Conclusion

- Endomorphism ring computation & pure isogeny problems are natural problems with some history
- Still, we need more classical and quantum cryptanalysis, especially on problem variants
- SIDH or CSIDH ? depends on two hypothetical threats
  - Improved torsion point attacks
    (or more attacks using further specificities in SIDH)
  - Devastating subexponential quantum attacks

Thanks !

- Questions ?

Bibiography

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- [D31] Deuring. *Die Typen der Multiplikatorenringe elliptischer Funktionenkörper*.
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- [W69] Waterhouse. *Abelian varieties over finite fields.*
- [KLPT14] Kohel-Lauter-Petit-Tignol. *On the quaternion ℓ-isogeny path problem.*
- [GPS17] Galbraith-Petit-Silva. *Identification Protocols and Signature Schemes Based on Supersingular Isogeny Problems*.
- [K96] Kohel. *Endomorphism rings of elliptic curves over finite fields.*

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- [DG16] Delfs-Galbraith. *Computing isogenies between supersingular elliptic curves over* $\mathbb{F}_p$.
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- [JS19] Jaques-Schanck. *Quantum cryptanalysis in the RAM model : Claw finding attacks on SIKE.*
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- [P17] Petit. *Faster Algorithms for Isogeny Problems Using Torsion Point Images.*
- [BS11] Bisson-Sutherland. *Computing the endomorphism ring of an ordinary elliptic curve over a finite field.*
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